Exercise 2

A Graphs of Rational Functions

1
Solution

(a) Given $y=\dfrac{2}{x+1}$

Equations of asymptotes

$x=-1$ is the vertical asymptote

$y=0$ is the horizontal asymptote.

Axial intercept : When $x=0$, $y=2$.

Determining the turning point

$\dfrac{dy}{dx}=-\dfrac{2}{(x+1)^2}$

Since $(x+1)^2>0$, for all real values of x, except -1

$\therefore\ \dfrac{dy}{dx}$ is always negative for all real values of x.

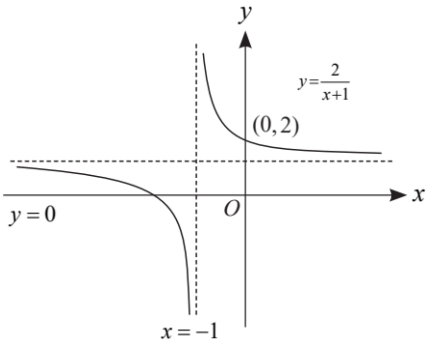
Thus there is no stationary point.

As $x\to\infty$, $y\to 0^+$

$x\to-\infty$, $y\to 0^-$

$x\to 1^+$, $y\to\infty$

$x\to 1^-$, $y\to-\infty$



(b) Given $y=3-\dfrac{2}{x-4}$.

Equations of asymptotes

$x=4$ is the vertical asymptote

$y=3$ is the horizontal asymptote.

Axial intercept : When $x=0$, $y=\dfrac{7}{2}$.

When $y=0$, $x=4\dfrac{2}{3}$.

Determining the turning point

$\dfrac{dy}{dx}=\dfrac{2}{(x-4)^2}$

$\because\ (x-4)^2>0$, for all real values of x, except 4

$\therefore\ \dfrac{dy}{dx}$ is always positive for all real values of x.

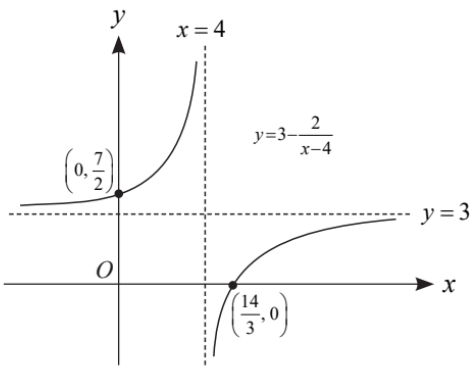
Thus there is no stationary point.

As $x\to+\infty$, $y\to 3^-$

$x\to-\infty$, $y\to 3^+$

$x\to 4^+$, $y\to-\infty$

$x\to 4^-$, $y\to\infty$



(c) Given $y=\dfrac{x+2}{x-1}$

By long division, $y=1+\dfrac{3}{x-1}$

Equations of asymptotes

$x=1$ is the vertical asymptote.

$y=1$ is the horizontal asymptote.

Axial intercept : When $x=0, y=-2$

When $y=0, x=-2$

Determining the turning point

$$\frac{dy}{dx}=-\frac{3}{(x-1)^2}$$

$\because\ (x-1)^2>0,$ for all real values of x, except 1.

$\therefore\ \dfrac{dy}{dx}$ is always negative for all real values of x.

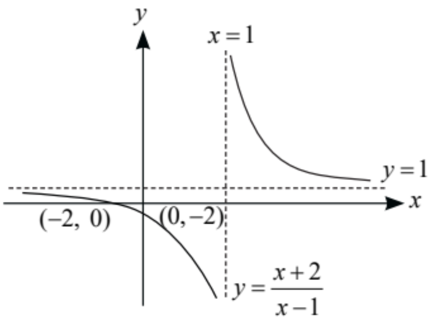
Thus there is no stationary point.

As $x\to\infty, y\to 1^+$

$x\to-\infty, y\to 1^-$

$x\to 1^+, y\to\infty$

$x\to 1^-, y\to-\infty$



(d) Given $y=\dfrac{2+3x}{2+x}$ ◃ perform long division

$$=3-\frac{4}{x+2}$$

Equations of asymptotes

$x=-2$ is the vertical asymptote.

$y=3$ is the horizontal asymptote.

Axial intercept : When $x=0, y=1$

When $y=0, x=-\dfrac{2}{3}$

Determining the turning point

$$\frac{dy}{dx}=\frac{4}{(x+2)^2}$$

$\because\ (x+2)^2>0,$ for all real values of x, except 2

$\therefore\ \dfrac{dy}{dx}$ is always positive for all real values of x.

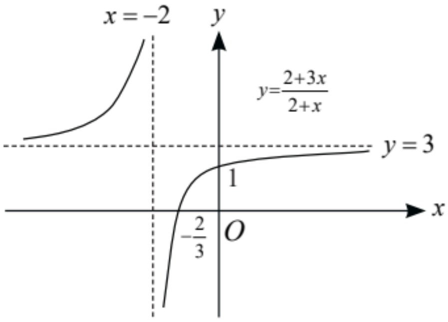
Thus there is no stationary point.

As $x\to\infty, y\to 3^-$

$x\to-\infty, y\to 3^+$

$x\to -2^+, y\to-\infty$

$x\to -2^-, y\to\infty$



(e) Given $y = x + 1 - \dfrac{2}{x}$

Equations of asymptotes

$x = 0$ is the horizontal asymptote.

$y = x+1$ is an oblique asymptote.

Axial intercept : When $y = 0,\ x = 1$ or $x = -2$

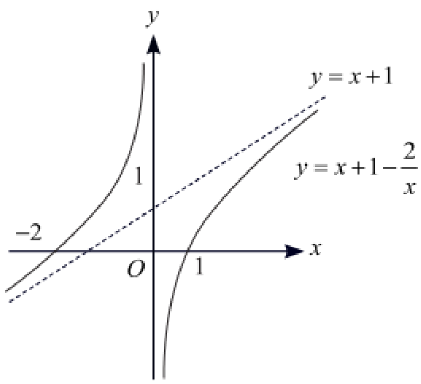
Determining the turning point

$\dfrac{dy}{dx} = 1 + \dfrac{2}{x^2}$

$\because \dfrac{2}{x^2} > 0,$ for all values of x, except $x = 0$.

$\therefore\ \dfrac{dy}{dx}$ is always positive.

Thus, there is no turning point.



As $\quad x \to \infty,\ y \to (x+1)^-$

$\quad x \to -\infty,\ y \to (x+1)^+$

$\quad x \to 0,\ y \to 0^-$

$\quad x \to 0,\ y \to 0^+$

(f) Given $y = \dfrac{3 + x - 2x^2}{1 + 2x}$ ◁ perform long division

$= -x + 1 + \dfrac{2}{2x+1}$

Equations of asymptotes

$y = -x + 1$ is an oblique asymptote.

$x = -\dfrac{1}{2}$ is a vertical asymptote.

Axial intercept : When $x = 0,\ y = 3$

When $y = 0\ \ x = \dfrac{3}{2},\ x = -1.$

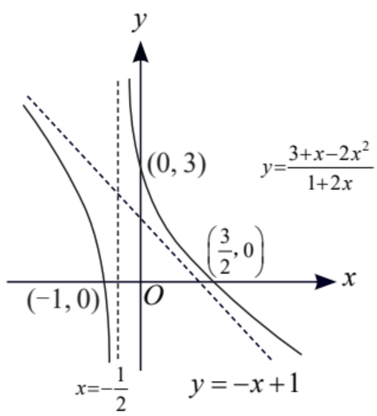
Determining the turning point

$\dfrac{dy}{dx} = -1 - \dfrac{4}{(2x+1)^2}$

$\because\ (2x+1)^2\ > 0,$ for all real values of x, except $x \neq -\dfrac{1}{2}$.

$\therefore\ \dfrac{dy}{dx}$ is always negative for all real values of x.

Thus there is no turning point.



As $\quad x \to \infty,\ y \to (-x+1)^+$

$\quad x \to -\infty,\ y \to (-x+1)^-$

$\quad x \to 1^+,\ y \to \infty$

$\quad x \to 1^-,\ y \to -\infty$

2
Solution

(a) $y=\dfrac{2x^2+cx+3}{x+2}$ ◁ long division

$=2x+c-4-\dfrac{11-2c}{x+2}$ .... (A)

Given that the oblique asymptote is $y=2x+4$,
by comparing the oblique asymptote in (A),
$c-4=4$
$\therefore\ c=8$

(b) Performing long division

$$
\begin{array}{r l}
 & \dfrac{p}{a}x+\dfrac{1}{a}\left(q-\dfrac{pb}{q}\right) \\ \cline{2-2}
ax+b \big) & px^2+qx+r \\
 & (-)\,px^2+\dfrac{pb}{a}x \\ \cline{2-2}
 & \left(q-\dfrac{pb}{a}\right)x+r \\
 & (-)\ \left(q-\dfrac{pb}{a}\right)x+\dfrac{b}{a}\left(q-\dfrac{pb}{a}\right) \\ \cline{2-2}
 & r-\dfrac{b}{a}\left(q-\dfrac{pb}{a}\right)
\end{array}
$$

$\therefore\ y=\dfrac{px^2+qx+r}{ax+b}$

$=\dfrac{p}{a}x+\dfrac{1}{a}\left(q-\dfrac{pb}{a}\right)+\dfrac{r-\dfrac{b}{a}\left(q-\dfrac{pb}{a}\right)}{ax+b}$

The oblique asymptote is $\dfrac{p}{a}x+\dfrac{1}{a}\left(q-\dfrac{pb}{a}\right)$ .... (1)

Given $y=ax+b$ is an asymptote to the curve, by comparing with (1).

i.e. $ax+b=\dfrac{p}{a}x+\dfrac{1}{a}\left(q-\dfrac{pb}{a}\right)$

Comparing coefficient of x

$\dfrac{p}{a}=a$

$p=a^2$

Comparing constant

$b=\dfrac{1}{a}\left(q-\dfrac{pb}{a}\right)$

Replace $p=a^2$

$b=\dfrac{1}{a}\left(q-\dfrac{a^2b}{a}\right)$

$b=\dfrac{q}{a}-\dfrac{a^2b}{a^2}$

$b=\dfrac{q}{a}-b$

$2ab=q$

From (1): $y = \frac{p}{a}x + \frac{1}{a}\left(q - \frac{pb}{a}\right) + \frac{r - \frac{b}{a}\left(q - \frac{pb}{a}\right)}{ax+b}$

Consider the quotient, $r - \frac{b}{a}\left(q - \frac{pb}{a}\right)$.

Replace q with $2ab$ and p wth a^2,

$= r - \frac{b}{a}\left(2ab - \frac{a^2}{a}b\right)$

$= r - \frac{b}{a}(2ab - ab)$

$= r - \frac{b}{a}(ab)$

$= r - b^2$

$\therefore\ r \neq b^2$

Therefore $r \neq b^2,\ q = 2ab,\ p = a^2$

(c) $y = ax + b + \frac{a+2b}{x-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{a+2b}{(x-1)^2}$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

i.e $\quad a - \frac{a+2b}{(x-1)^2} = 0$

$$a = \frac{a+2b}{(x-1)^2}$$

$$(x-1)^2 = \frac{a+2b}{a}$$

$$(x-1) = \sqrt{\frac{a+2b}{a}} \text{ ....} (1)$$

Since C has no stationary point, equation (1) does not have real roots.

$$\frac{a+2b}{a} < 0$$

$$a + 2b < 0$$

$$a < -2b$$

(d) Using long division

$$\begin{array}{r} x+a-b \\ x+b\,\overline{)\,x^2+ax+b^2} \\ -\underline{x^2+bx} \\ (a-b)x+b^2 \\ -\underline{(a-b)x+b(a-b)} \\ 2b^2-ab \end{array}$$

$\therefore\ \mathrm{f}(x)=x+a-b+\dfrac{2b^2-ab}{x+b}$

The equation of the oblique asymptote is $y=x+a-b$.

If $a=2b$,

$$y=x+2b-b+\frac{2b^2-2b^2}{x+b} \quad \triangleleft \text{ replace } a=2b$$
$$=x+b$$

$\therefore$ the graph $y=\mathrm{f}(x)$ becomes a straight line $y=x+b$.

(e) $\mathrm{f}(x)=\dfrac{ax+b}{cx+d}$

$$\mathrm{f}'(x)=\frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$$
$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$=\frac{ad-bc}{(cx+d)^2}$$

When $ad-bc=0$, $\mathrm{f}'(x)=\dfrac{0}{(cx+d)^2}=0$.

The gradient of the curve at all points is 0. ‘

Hence, the graph is a horizontal line.

3.
Solution

(a) Given $y = \dfrac{(x+2)^2}{x+1}$ ⊲ long division

$$= x + 3 + \frac{1}{x+1} \text{ .... (1) (verified)}$$

From (1), $A = 3$ and $B = 1$

The equations of the asymptotes are $y = x + 3$ and $x = -1$.

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - (x+1)^{-2} \text{ .... (2)}$$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$1 - (x+1)^{-2} = 0$$
$$\frac{1}{(x+1)^2} = 1$$
$$(x+1)^2 = 1$$
$$(x+1) = \pm 1$$
$$x = 0 \quad \text{or} \quad -2$$

Substituting $x = 0$ into (1): $\therefore\ y = 4$
Substituting $x = -2$ into (1): $\therefore\ y = 0$

The stationary points are $(0,\ 4)$ and $(-2,\ 0)$.

Differentiate (1) with respect to x

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2(x+1)^{-3}$$

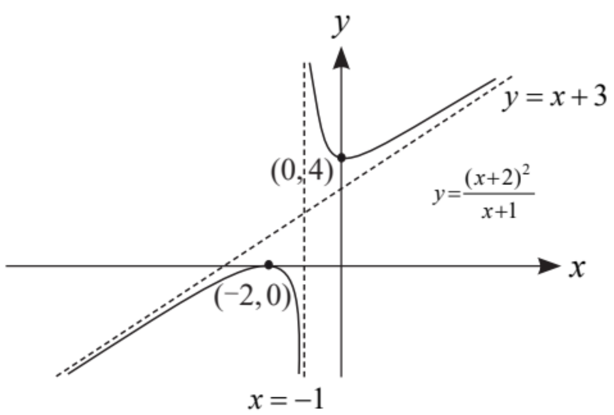
When $x = 0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$

$\therefore\ (0,\ 4)$ is a minimum point.

When $x = -2$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$

$\therefore\ (-2,\ 0)$ is a maximum point.

When $x = 0$, $y = 4$. $\therefore\ y$-intercept $= 4$
When $y = 0$, $x = -2$. $\therefore\ y$-intercept $= 4$



(c) Let $y = k$ .... (3)

Substituting (3) into (1)

$$k = \frac{x^2 + 4x + 4}{x + 1}$$

$$k(x+1) = x^2 + 4x + 4$$

$x^2 + 4x - kx + 4 - k = 0$ .... (4)

For the line does not cut the curve, the discriminant of (4) must be negative, i.e. $b^2 - 4ac < 0$.

$$(4-k)^2 - 4(1)(4-k) < 0$$

$$(4-k)(4-k-4) < 0$$

$$-k(4-k) < 0$$

$$k(k-4) < 0$$

$$0 < k < 4$$

The solution set of k is $\{k : k \in \mathbb{R}, 0 < k < 4\}$.

Alternative Method

For the line to cut the curve, the discriminant of (4) must be real.

i.e. $b^2 - 4ac \geq 0$

$$(4-k)^2 - 4(1)(4-k) \geq 0$$

$$k(k-4) \geq 0$$

$$k \geq 4 \quad \text{or} \quad k \leq 0$$

So for the line does not intersect the curve, then the set of values of k is $\{k : k \in \mathbb{R}, 0 < k < 4\}$.

4
Solution

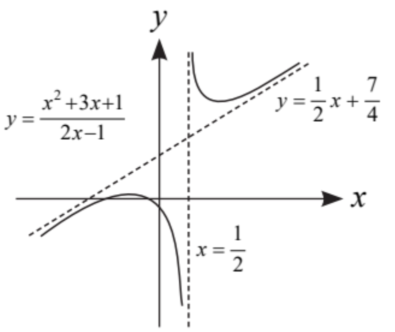
(a) Given $y=\dfrac{x^2+3x+1}{2x-1}$

By long division,

$$=\frac{1}{2}x+\frac{7}{4}+\frac{11}{4(2x-1)}$$

Vertical asymptote : $x=\dfrac{1}{2}$

oblique asymptote : $y=\dfrac{1}{2}x+\dfrac{7}{4}$



x-intercepts: $(-2.62, 0)$,
$(-0.382, 0)$
y-intercept: $(0, -1)$
Max pt: $(-1.16, 0.342)$
Min pt: $(2.16, 3.66)$

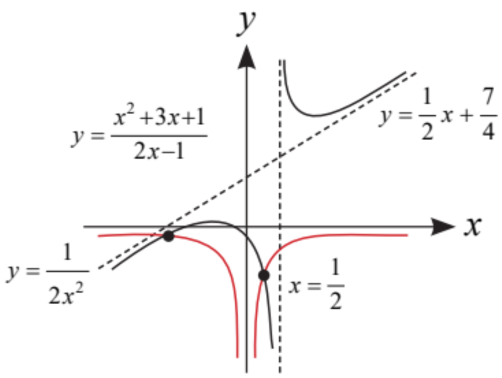
(b) $2x^4+6x^3+2x^2+2x-1=0$

$$2x^2(x^2+3x+1)=-(2x-1)$$

$$\frac{x^2+3x+1}{2x-1}=-\frac{1}{2x^2}$$

$\therefore\ y=-\dfrac{1}{2x^2}$

Add the graph $y=-\dfrac{1}{2x^2}$ on the same diagram.



Refer to the above diagram. The graph of $y=-\dfrac{1}{2x^2}$ intersects the graph of $y=\dfrac{x^2+3x+1}{2x-1}$ at 2 points.

$\therefore$ there are 2 real roots of the equation $2x^4+6x^3+2x^3+2x-1=0$.

5
Solution

(a) Let $y = \text{g}(x)$

$$\therefore \quad \frac{x^2 - x + 2}{x - 1} = y$$

$$xy - y = x^2 - x + 2$$

$$x^2 + (-1 - y)x + (2 + y) = 0$$

For $x \in \mathbb{R}$, discriminant ≥ 0

$$(-1 - y)^2 - 4(1)(2 + y) \geq 0$$

$$1 + 2y + y^2 - 8 - 4y \geq 0$$

$$y^2 - 2y - 7 \geq 0$$

Let $y^2 - 2y - 7 = 0$

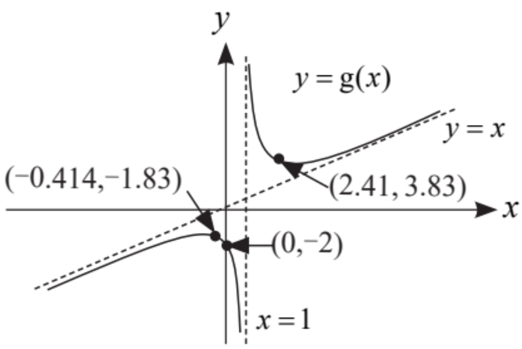
$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2(1)}$$

$$y = \frac{2 \pm \sqrt{32}}{2}$$

$$y = 1 \pm 2\sqrt{2}$$

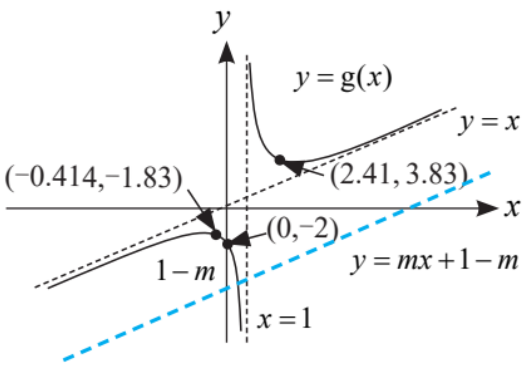
$\therefore$ Range of g is $\left(-\infty, 1 - 2\sqrt{2}\right] \cup \left[1 + 2\sqrt{2}, \infty\right)$.

(b) Graph of $\text{g}(x) = \dfrac{x^2 - x + 2}{x - 1}$.



(c) Add the line $y = mx + 1 - m$ to the same diagram.

$\therefore\ m > 1$



Learning point:

Note that the line $y = mx + 1 - m$ is parallel to the equation of asymptote, $y = x$.

If $m > 1$, then the line $y = mx + 1 - m$ will become **steeper** and hence this line will cut the curve at two points.

If $m < 1$, then the line $y = mx + 1 - m$ will become **flatter** and hence this line will not cut the curve at two points.

6
Solution

(a) Let the denominator of $y = \dfrac{ax^2 + bx + a}{x^2 - b}$ be zero.

i.e. $\qquad x^2 - b = 0$

$(x + \sqrt{b})(x - \sqrt{b}) = 0$

So, $x = \sqrt{b}$ and $x = -\sqrt{b}$ are vertical asymptotes.

Given $x = 2,\ \sqrt{b} = 2.$

Hence $\qquad b = 4.$

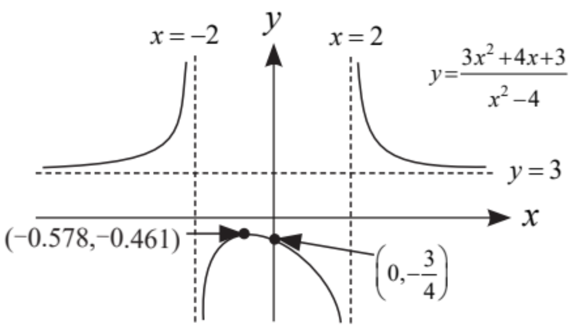
$$y = \frac{ax^2 + bx + a}{x^2 - b}$$
$$= a + \frac{bx + ab + a}{x^2 - b}$$

So, $y = a$ is a vertical asymptote.

Given that $y = 3,$ hence, $a = 3.$

$\therefore\ a = 3$ and $b = 4$

(b) Graph of $y = \dfrac{3x^2 + 4x + 3}{x^2 - 4}$



7
Solution

(a) Let the denominator of $y=\dfrac{ax^2+2x+b}{x+c}$ be zero.

i.e. $x+c=0$

$x=-c$

So, $x=-c$ is the vertical asymptote.

Given $x=-5$ is the vertical asymptote,

$\therefore \quad c=5.$

$\therefore\ y=\dfrac{ax^2+2x+b}{x+5}$

When $x=0,\ y=\dfrac{1}{5}$,

$\dfrac{1}{5}=\dfrac{a(0)^2+2(0)+b}{0+5}$

$b=1.$

$\therefore\ y=\dfrac{ax^2+2x+1}{x+5}$

$=ax+(2-5a)+\dfrac{25a-9}{x+5}$

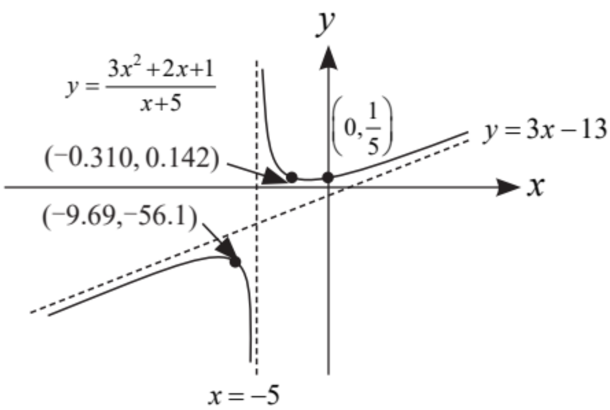
Given $y=3x-13$ is an oblique asymptote,

by comparing with $y=3x-13$, $a=3$.

$\therefore\ y=\dfrac{3x^2+2x+1}{x+5}$

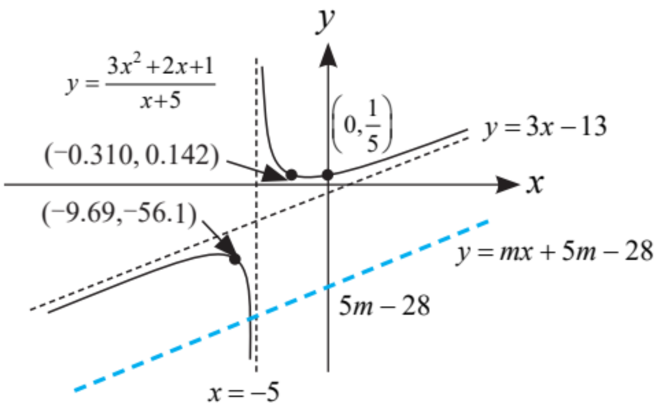
$\therefore\ \ a=3,\ b=1$ and $c=5$

(b) Graph of $y=\dfrac{3x^2+2x+1}{x+5}$.



(c) Given $3x^2+2x+1=(x+5)(mx+5m-28)$

$$\frac{3x^2+2x+1}{x+5}=mx+5m-28$$



(i) For $3x^2+2x+1=(x+5)(mx+5m-28)$ to have 2 real roots,

$\therefore\ m>3$

(ii) For $3x^2+2x+1=(x+5)(mx+5m-28)$ to have no real roots,

$m\leq 3$

8
Solution

(a) Given $y=\dfrac{x^2+ax-4}{x+b}$. .... (1)

The vertical asymptote of $y=\dfrac{x^2+ax-4}{x+b}$ is $x=-b$.

Given in the question $x=1$ is a vertical asymptote, by comparing with $x=-b$.

$b=-1$.

Given $y=x-2$ is an oblique asymptote, we can express the equation as

$y=x-2+\dfrac{k}{x-1}$, where k is a constant. .... (2)

Method 1

$$y=\frac{x^2+ax-4}{x-1} \quad \triangleleft \text{ long division}$$
$$=x+(a+1)+\frac{a-3}{x-1} \text{ .... } (3)$$

Comparing (2) and (3)

$$a+1=-2$$
$$a=-3$$

Method 2

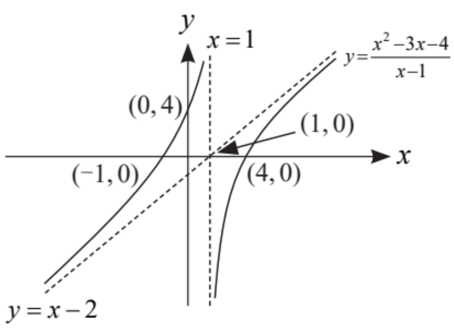
$$y=x-2+\frac{k}{x-1} \quad \triangleleft \text{ express (2) as a single fraction}$$
$$=\frac{x^2-3x+2+k}{x-1} \text{ .... } (4)$$

Comparing (1) and (4)

$a=-3$.

$\therefore\ a=-3$ and $b=-1$

Sketch the curve C as shown below.



(b) $x^2-2x-20+21\left(\dfrac{x^2-3x-4}{x-1}\right)^2=0$

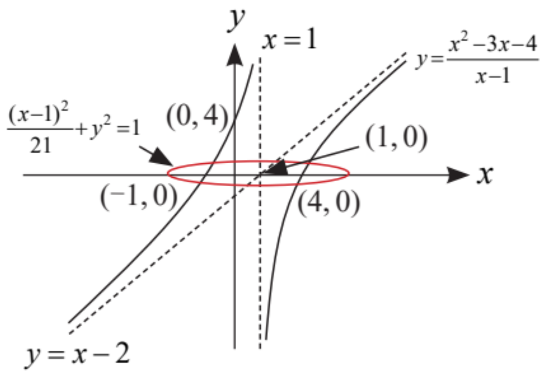
$$(x-1)^2-1-20+21\left(\frac{x^2-3x-4}{x-1}\right)^2=0$$

$$(x-1)^2+21\left(\frac{x^2-3x-4}{x-1}\right)^2=21$$

$$\frac{(x-1)^2}{21}+\left(\frac{x^2-3x-4}{x-1}\right)^2=1$$

$$\frac{(x-1)^2}{21}+(y)^2=1$$

$$\frac{(x-1)^2}{(\sqrt{21})^2}+\frac{y^2}{1^2}=1$$



The added graph is an ellipse centred at $(1, 0)$ with horizontal axis length $\sqrt{21}$ and vertical axis length 1.

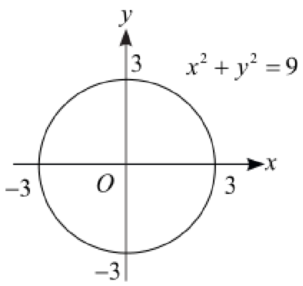
From graph, the ellipse intersects curve C at 4 distinct points, therefore it has <u>4 real distinct roots</u>.

Exercise 2

B Conics

9.
Solution

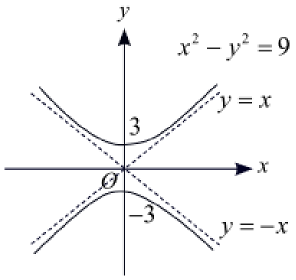
(a) The graph of $x^2 + y^2 = 9$.



Learning point:

The graph is a circle centred at (0, 0) with radius 3.

(b) The graph of $y^2 - x^2 = 9$.

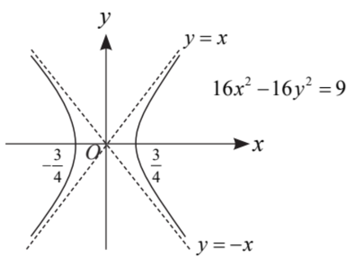


Learning point:

The graph is a hyperbola centred at (0, 0) and opens up and down. The graph has two equations of the asymptotes $y = x$ and $y = -x$.

(c) Rewrite the equation $16x^2 - 16y^2 = 9$ as $\dfrac{x^2}{\left(\frac{3}{4}\right)^2} - \dfrac{y^2}{\left(\frac{3}{4}\right)^2} = 1$.

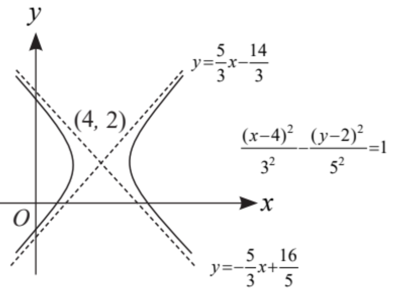
The graph of $16x^2 - 16y^2 = 9$.



Learning point:

The graph is a hyperbola centred at (0, 0) and opens left and right. The graph has two equations of the asymptotes $y = x$ and $y = -x$.

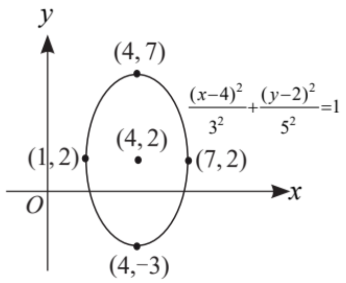
(d) The graph of $\dfrac{(x-4)^2}{3^2}-\dfrac{(y-2)^2}{5^2}=1$



Learning point:

The graph is a hyperbola centred at (4, 2) and opens left and right. The graph has two equations of the asymptotes $y=\dfrac{5}{3}x-\dfrac{5}{3}$ and $y=-\dfrac{5}{3}x+\dfrac{16}{5}$.

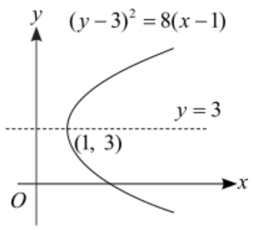
(e) The graph of $\dfrac{(x-4)^2}{3^2}+\dfrac{(y-2)^2}{5^2}=1$



Learning point:

The graph is an ellipse centred at (0, 0) with horizontal axis length 3 and vertical axis length 5.

(f) The graph of $(y-3)^2=8(x-1)$



Learning point:

The graph is a parabola. It opens on the right with vertex (1, 3) and .is symmetrical about y = 3.

10
Solution

(a)

$$\frac{1}{4}x^2 - 3x + \frac{1}{4}y^2 + y = 1$$

$$\frac{1}{4}(x^2 - 12x) + \frac{1}{4}(y^2 + 4y) = 1 \quad \triangleleft \text{ completing the square}$$

$$\frac{1}{4}\left[(x-6)^2 - 6^2\right] + \frac{1}{4}\left[(y+2)^2 - 2^2\right] = 1$$

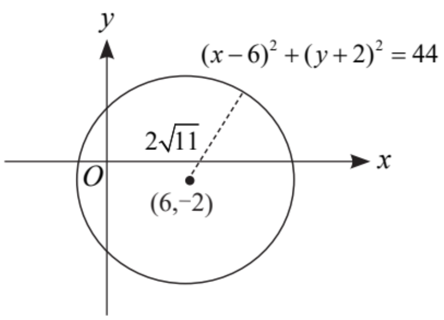
$$\frac{1}{4}(x-6)^2 - 9 + \frac{1}{4}(y+2)^2 - 1 = 1$$

$$\frac{1}{4}(x-6)^2 + \frac{1}{4}(y+2)^2 = 11$$

$$(x-6)^2 + (y+2)^2 = 44$$

$$(x-6)^2 + [y-(-2)]^2 = 44$$

$$(x-6)^2 + [y-(-2)]^2 = (2\sqrt{11})^2$$



The equation shows an equation of a circle.
Centre of circle = $(6, -2)$
Radius of circle = $2\sqrt{11}$.

(b) $9x^2 + 18x - 4y^2 + 4y = 28$

$9(x^2 + 2x) - 4(y^2 - y) = 28$

$9\left[(x+1)^2 - 1^2\right] - 4\left[\left(y - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right] = 28$ ◁ completing the square

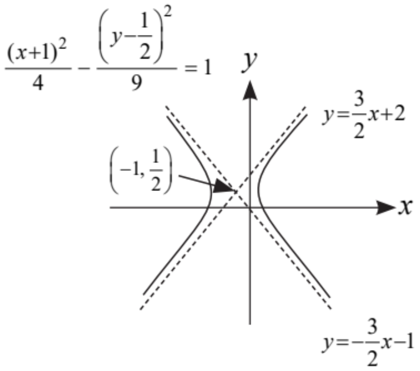
$9(x+1)^2 - 9 - 4\left(y - \frac{1}{2}\right)^2 + 1 = 28$

$9(x+1)^2 - 4\left(y - \frac{1}{2}\right)^2 = 9 - 1 + 28$

$9(x+1)^2 - 4\left(y - \frac{1}{2}\right)^2 = 36$

$\frac{9(x+1)^2}{36} - \frac{4(y-\frac{1}{2})^2}{36} = 1$

$\frac{(x+1)^2}{4} - \frac{(y-\frac{1}{2})^2}{9} = 1$



The equation represents a hyperbola.

Centre of hyperbola $= \left(-1, \frac{1}{2}\right)$

Gradient of asymptotes $= \pm\frac{3}{2}$

Equations of asymptotes : $y = \frac{3}{2}x + 2$ and $y = -\frac{3}{2}x - 1$.

(c) $4x^2 + y^2 + 16x + 7 = 0$

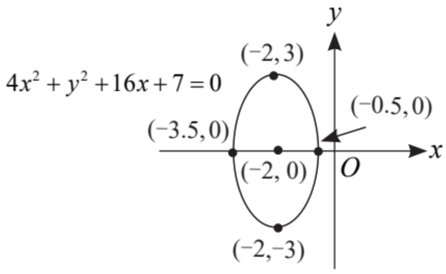
$4(x^2 + 4x) + y^2 + 7 = 0$

$4\left[(x+2)^2 - 2^2\right] + y^2 + 7 = 0$

$4(x+2)^2 - 16 + y^2 + 7 = 0$

$4(x+2)^2 + y^2 = 9$

$\dfrac{(x+2)^2}{\left(\frac{3}{2}\right)^2} + \dfrac{y^2}{3^2} = 1$



The equation respresents an ellipse.
Centre of ellipse : $(-2,\ 0)$.
Lines of symmetry: $x = -2$ and $y = 0$.

11.
Solution

$$2x^2+3y^2+6ax+6by=0$$

$$2\left(x^2+3ax+\frac{9}{4}a^2\right)+3\left(y^2+2by+b^2\right)-\frac{9}{2}a^2-3b^2=0$$

$$2\left(x+\frac{3}{2}a\right)^2+3(y+b)^2=\frac{9}{2}a^2+3b^2 \text{ .... (1)}$$

The centre of the ellipse is $\left(-\frac{3}{2}a, b\right)$

From the diagram, the centre of ellipse is (0, 1).

By comparing the coordinates $\left(-\frac{3}{2}a, b\right)$ and (0, 1).

$$-\frac{3}{2}a=0 \text{ and } -b=1.$$

$\therefore\ a=0$ and $b=-1$

Substitute a and b into (1):

$$2\left(x+\frac{3}{2}a\right)^2+3\left(y+b\right)^2=\frac{9}{2}a^2+3b^2$$

$$2x^2+3(y-1)^2=3$$

$$\frac{x^2}{\frac{3}{2}}+(y-1)^2=1$$

$$\frac{x^2}{\left(\sqrt{\frac{3}{2}}\right)^2}+\frac{(y-1)^2}{1^2}=1$$

The ellipse has the horizontal axis length $\sqrt{\frac{3}{2}}$.

$\therefore\ k=\sqrt{\frac{3}{2}}$.

12.
Solution

(a) Given $\dfrac{(y+5)^2}{b^2}-\dfrac{(x-1)^2}{a^2}=1$

From the equation, the gradients of the asymptotes $=\pm\dfrac{b}{a}$.

Given that the oblique asymptote is $y=2x-7$. Its gradient is 2.

So $\dfrac{b}{a}=2$. .... (1)

The hyperbola has a minimum point $(1,-1)$. Substitute the coordinates into the equation.

$$\frac{(-1+5)^2}{b^2}-\frac{(1-1)^2}{a^2}=1$$

$$b^2=4^2$$

$$b=4 \quad (b>0) \quad \text{(Shown)}$$

Substituting $b=4$ into (1).

$$\frac{4}{a}=2.$$

$\therefore\ a=2$ (Shown)

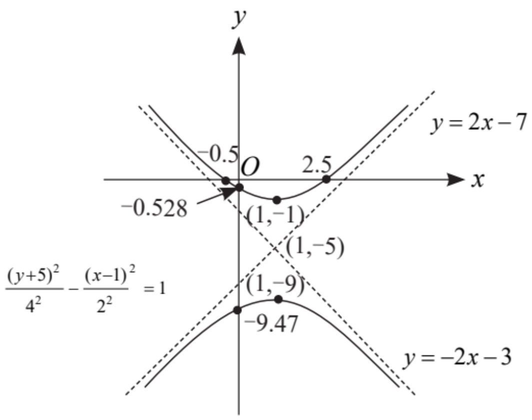
(b) Since centre of the hyperbola is $(1,-5)$, and the gradient of the other asymptote is -2, so its equation of other oblique asymptote is

$$y-(-5)=-2(x-1)$$

$$y+5=-2x+2$$

Equation of other oblique asymptote is $y=-2x-3$

(c)



13.
Solution

(a) Given $9x^2 - ay^2 - 36x - 2aby + 9a - ab^2 + 36 = 0$ .... (1)

C passes through $(2, -2)$. Substitute $x = 2$ and $y = -2$ into the equation.

$$9(2)^2 - a(-2)^2 - 36(2) - 2ab(-2) + 9a - ab^2 + 36 = 0$$
$$36 - 4a - 72 + 4ab + 9a - ab^2 + 36 = 0$$
$$4ab + 5a - ab^2 = 0$$
$$-a(b^2 - 4b - 5) = 0$$
$$b^2 - 4b - 5 = 0, \quad a \neq 0$$
$$(b-5)(b+1) = 0$$
$$b = 5 \quad \text{or} \quad -1 \text{ (rejected since } b > 0)$$

Substitute $b = 5$ into (1).

$$9x^2 - ay^2 - 36x - 10ay + 9a - 25a + 36 = 0$$
$$9(x^2 - 4x + 4) - a(y^2 + 10y + 25) + 9a = 0 \quad \triangleleft \text{ completing the square}$$
$$9(x-2)^2 - a(y+5)^2 + 9a = 0$$
$$a(y+5)^2 - 9(x-2)^2 - 9a = 0$$
$$a(y+5)^2 - 9(x-2)^2 = 9a \quad \triangleleft \text{ divide } 9a \text{ throughout}$$
$$\frac{(y+5)^2}{9} - \frac{(x-2)^2}{a} = 1$$
$$\frac{(y+5)^2}{3^2} - \frac{(x-2)^2}{(\sqrt{a})^2} = 1$$

From the equation, the gradients of the asymptotes $= \pm\dfrac{3}{\sqrt{a}}$.

Given that the oblique asymptote is $y = \dfrac{3}{2}x - 8$. Its gradient is $\dfrac{3}{2}$.

So $\dfrac{3}{\sqrt{a}} = \dfrac{3}{2}$

$a = 4$

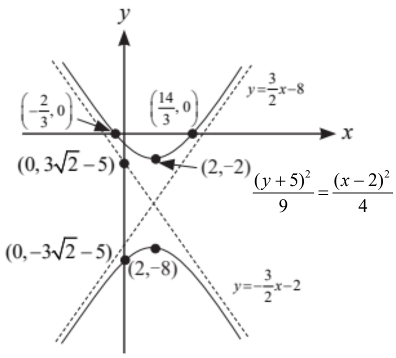
$\therefore\ a = 4$ and $b = 5$

(b) Equations of the asymptotes

$$y + 5 = \pm\frac{3}{2}(x-2)$$
$$y = \frac{3}{2}x - 8 \quad \text{or} \quad y = -\frac{3}{2}x - 2$$

Equation of the other asymptote is $y = -\dfrac{3}{2}x - 2$

(c)

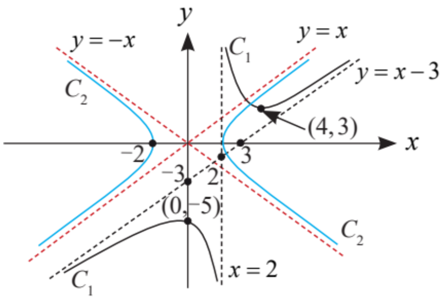


14
Solution

(a) $C_1 : y = \dfrac{x^2 - 5x + 10}{x - 2}$ .... (1)

$= x - 3 + \dfrac{4}{x-2}$

$C_2 : x^2 - y^2 = 4$ .... (2)



(b) The x-coordinate of the point of intersection of C_1 and $C_2 \Rightarrow$ solve C_1 and C_2 simultaneously.

Substituting (1) into (2).

$$x^2 - \left(\frac{x^2 - 5x + 10}{x - 2}\right)^2 = 4$$

$$(x^2 - 4) = \frac{(x^2 - 5x + 10)^2}{(x-2)^2}$$

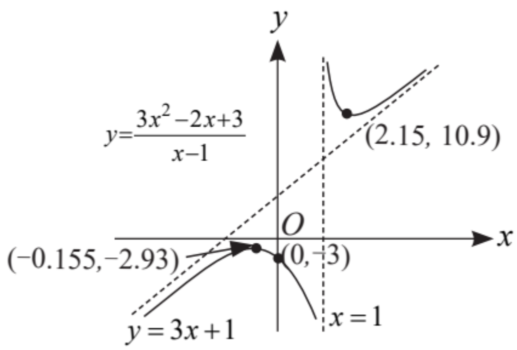
$$(x^2 - 4)(x-2)^2 = (x^2 - 5x + 10)^2 \quad \text{(Shown)}$$

(c) Use GC, $x = 3.66$ (correct to 3 sf)

15
Solution

(a) $y = \dfrac{3x^2 - 2x + 3}{x - 1}$

$= 3x + 1 + \dfrac{4}{x-1}$



(b) The range of b is $0 < b \le 3$

Learning point:

Given C_2 : $(x-1)^2 - \dfrac{(y-4)^2}{b^2} = 1$

C_2 is a horizontal hyperbola with centre (1, 4).

Asymptotes: $y = \pm b(x-1) + 4$

For no point of intersection between C_1 and C_2, the asymptote of hyperbola C_2 must be the same as the oblique asmptote of C_1. or less steep than the oblique asmptote of C_1.

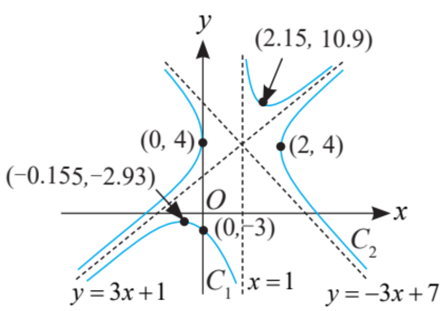
Note that the gradient of oblique asymptote of C_1 is 3.

$\therefore\ 0 < b \le 3$

(c) For $b = 3$,

Asymptotes: $y = \pm 3(x-1) + 4$

$y = 3x + 1$ and $y = -3x + 7$



16.
Solution

(a) $9y^2+36y-4x^2+8x-4=0$

$9(y^2+4y)-4(x^2-2x)-4=0$

$9(y^2+4y+4)-36-4(x^2-2x+1)+4-4=0$

$9(y+2)^2-4(x-1)^2=36$

$\dfrac{(y+2)^2}{2^2}-\dfrac{(x-1)^2}{3^2}=1$ (Shown)

(b) Lines of symmetry are $x=1$ and $y=-2$.

(c) To find y-intercepts, let $x=0$,

$$\frac{(y+2)^2}{2^2}-\frac{(-1)^2}{3^2}=1$$

$$\frac{(y+2)^2}{2^2}=\frac{10}{9}$$

$$(y+2)^2=\frac{40}{9}$$

$$y+2=\pm\sqrt{\frac{40}{9}}$$

$$y=\pm\frac{2}{3}\sqrt{10}-2$$

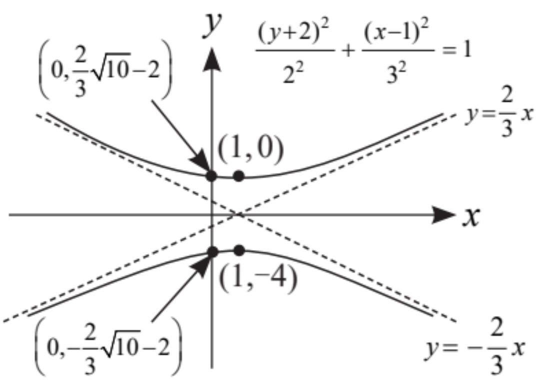
To find asymptotes, let $\dfrac{(y+2)^2}{2^2}-\dfrac{(x-1)^2}{3^2}=0$.

$$(y+2)^2=\frac{2^2(x-1)^2}{3}$$

$$y+2=\pm\frac{2}{3}(x-1)$$

$$y=\frac{2}{3}(x-1)-2 \text{ or } y=-\frac{2}{3}(x-1)-2$$

$$\therefore y=\frac{2}{3}x-\frac{8}{3} \text{ or } y=-\frac{2}{3}x-\frac{4}{3}$$



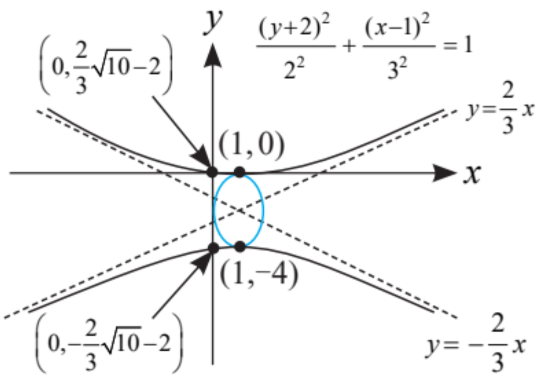
(d) $2m(x-1)^2+(y+2)^2=2m$

$$(x-1)^2+\frac{(y+2)^2}{2m}=1$$

$$\frac{(x-1)^2}{1^2}+\frac{(y+2)^2}{(\sqrt{2m})^2}=1$$

The above equation represents an ellipse centred at $(1,-2)$ with horizontal axis length 1 and vertical axis length $\sqrt{2}m$.

Add the ellipse to the diagram, as shown.



For H and J intersect at least twice, $\sqrt{2m}\geq 2$

$\therefore$ the range of m is $m\geq 2$

Exercise 2

C Parametric Equations

17.
Solution

(a) $x = 2t^2$ .... (1)

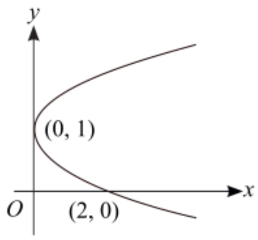
$y = 1 - t$ .... (2)

From (2): $t = 1 - y$ .... (3)

Substituting (3) into (1).

$\therefore$ the cartesian equation is $x = 2(1-y)^2$.

The curve of $x = 2t^2$, $y = 1 - t$.



(b) $x=-4t-\frac{1}{t}$ ◁ multiply $\frac{1}{2}$ on both sides

$$\frac{1}{2}x=-2t-\frac{1}{2t} \quad \text{.................................} \ (1)$$

$$y=3t+\frac{1}{2t} \quad \text{....................................} \ (2)$$

$(2)+(3)$:

$$y+\frac{1}{2}x=t$$

$$t=\frac{2y+x}{2} \quad \text{....................................} \ (3)$$

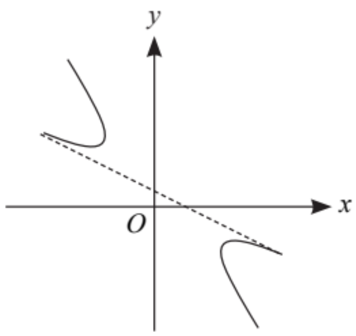
Substituting (4) into (1)

$$x=-4\left(\frac{2y+x}{2}\right)-\frac{1}{\left(\frac{2y+x}{2}\right)}$$

$$x=-2(2y+x)-\frac{2}{2y+x}$$

$\therefore$ the cartesian equation is $x(2y+x)=-2(2y+x)^2-2.$

The curve of $x=-4t-\frac{1}{t},\ y=3t+\frac{1}{2t}.$



(c) $x = \sec\theta$ .... (1)

$y = \tan\,\theta$ .... (2)

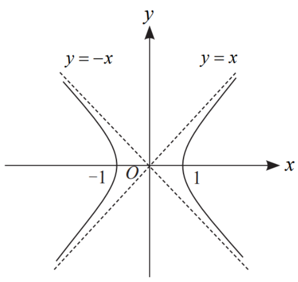
Using Trigogometric Identities, $\tan^2\theta + 1 = \sec^2\theta$ .... (3)

Substituting (1) and (2) into (3).

$$(y)^2 + 1 = (x)^2$$

$\therefore$ the cartesian equation is $y^2 + 1 = x^2$

The curve of $x = \sec\theta,\; y = \tan\theta.$



(d) $x = 3\cos t$ .... (1)

$y = 2\text{sint}$ .... (2)

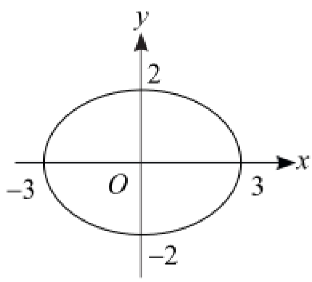
Using Trigogometric identities, $\sin^2 t + \cos^2 t = 1$............ (3)

Substituting (1) and (2) into (3).

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$$

$\therefore$ the cartesian equation is $\frac{y^2}{4} + \frac{x^2}{9} = 1$

The curve of $x = 3\cos t,\; y = 2\sin t.$



18
Solution

(a) $x+y$

$= (\cos\theta - \sin^2\theta) + (2\cos\theta + \sin^2\theta)$

$= 3\cos\theta$ .... (1)

$y-2x$

$= (2\cos\theta + \sin^2\theta.) - 2(\cos\theta - \sin^2\theta)$

$= \sin^2\theta$ .... (2)

From (1): $x+y = 3\cos\theta$

$$\frac{x+y}{3} = \cos\theta \quad \triangleleft \text{ square both sides}$$

$$\left(\frac{x+y}{3}\right)^2 = \cos^2\theta \text{ .... (1)}$$

From (2): $y-2x = 3\sin^2\theta$

$$\frac{y-2x}{3} = \sin^2\theta \text{ .... (3)}$$

Using identity $\sin^2\theta + \cos^2\theta = 1$ .... (4)

Substituting (1) and (3) into (4).

$$\frac{y-2x}{3} + \left(\frac{x+y}{3}\right)^2 = 1$$

$$\frac{y-2x}{3} + \frac{(x+y)^2}{9} = 1 \quad \triangleleft \text{ multiply 9 throughout}$$

$$3(y-2x) + (x+y)^2 = 9$$

$\therefore$ the Cartesian equation of the curve is $3(y-2x)+(x+y)^2 = 9$.

(b) Given $2y + x^2 = 4$ .... (1)

and $x = 2\cos t$ .... (2)

Substituting (1) into (2)

$$2y + (2\cos t)^2 = 4$$

$$2y + 4\cos^2 t = 4$$

$$y + 2\cos^2 t = 2$$

$$y = 2 - 2\cos^2 t$$

$$y = 2(1-\cos^2 t)$$

$\therefore$ the corresponding parametric expression for y is $y = 2(1-\cos^2 t)$.

19
Solution

(a) Let $x = \dfrac{2}{t}$ .... (1)

$y = 3t^2$ .... (2)

Substituting $t = 2$ into (1) and (2): $x = 1$ and $y = 3(2)^2 = 12$

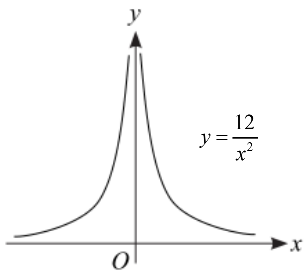
$\therefore$ the coordinates of the point P is $(1, 12)$.

(b) From (1): $t = \dfrac{2}{x}$......................... (3)

Substituting (3) into (2) gives $y = \dfrac{12}{x^2}$.

$\therefore$ the cartesian equation of the curve is $y = \dfrac{12}{x^2}$.

The graph of $y = \dfrac{12}{x^2}$.



20.
Solution

(a) $x^2 = \frac{1}{4}(e^{6t} + 4 + 4e^{-6t})$ .... (1)

$y^2 = \frac{1}{4}(e^{6t} - 4 + 4e^{-6t})$ .... (2)

Take $(1) - (2)$:

$x^2 - y^2 = \frac{1}{4}(e^{6t} + 4 + 4e^{-6t}) - \frac{1}{4}(e^{6t} - 4 + 4e^{-6t})$

$\therefore$ the cartesian equation of C is $x^2 - y^2 = 2$. .... (3)

Restriction on the values of x:

From (1): $e^{3t} > 0$, $e^{-3t} > 0$ for all real values of x.

$\therefore \frac{1}{2}(e^{3t} + 2e^{-3t}) > 0$. Hence, $x > 0$.

From (3): $x^2 - y^2 = 2$ $\lhd$ express x in terms of y

$y = \pm\sqrt{x^2 - 2}$

For y to exist, $x^2 - 2$ must be equal or more than zero.

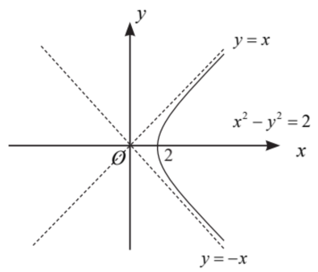
i.e. $x^2 - 2 \geq 0$

$(x - \sqrt{2})(x + \sqrt{2}) \geq 0$

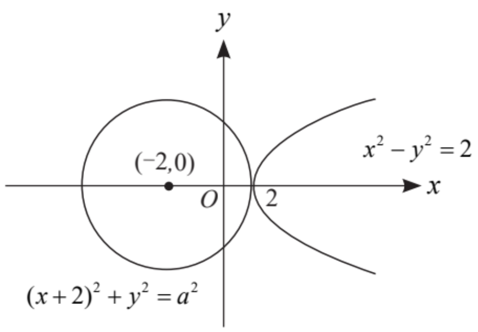
$\therefore x \leq -\sqrt{2}$ (rejected since $x > 0$) or $x \geq \sqrt{2}$

Thus $x \geq \sqrt{2}$

(b)



(c)



Given $x^2 + y^2 + 4x = a^2 - 4$

$x^2 + y^2 + 4x + 4 = a^2$

$x^2 + 4x + 4 + y^2 = a^2$

$(x+2)^2 + (y-0)^2 = a^2$

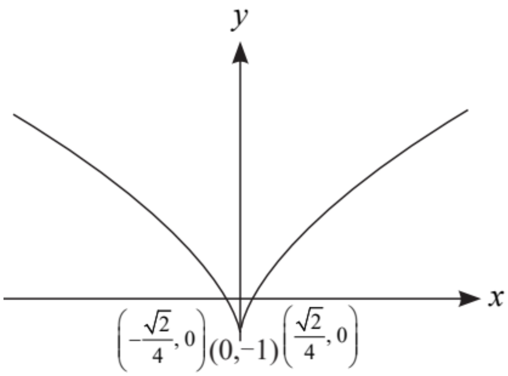
The equation represents a circle with centred $(-2,\ 0)$ and radius a.

For the circle never intersect C, the radius must not greater than 4.

$\therefore\ 0 < a < 4$

21.
Solution

(a) The graph of $x = t^3$, $y = 2t^2 - 1$ for $t \in \mathbb{R}$.



(b) Substitute $x = t^3$, $y = 2t^2 - 1$ into $y = -x - 1$:

$$2t^2 - 1 = -t^3 - 1$$
$$t^3 + 2t^2 = 0$$
$$t^2(t+2) = 0$$

$\therefore\ t = 0$ or $t = -2$

When $t = 0$, $x = 0$ and $y = -1$
$\therefore\ A(0, -1)$

When $t = -2$, $x = -8$ and $y = 7$
$\therefore\ B(-8, -7)$

Length of AB
$= \sqrt{(0-(-8))^2 + (-1-7)^2}$
$= \sqrt{128}$
$= 8\sqrt{2}$, where $k = 8$

Alternative Method

Converting the parametric equation into Cartesian equation.

Form $x = t^3$ becomes $t = x^{\frac{1}{3}}$ .... (1)

Substitute (1) into $y = 2t^2 - 1$

Thus $y = 2x^{\frac{2}{3}} - 1$......... (2)

Equate (2) with $y = -x - 1$ to find the points of intersection.

$\therefore \quad 2x^{\frac{2}{3}} - 1 = -x - 1$

$x^{\frac{2}{3}}\left(2 + x^{\frac{1}{3}}\right) = 0$

$x^{\frac{2}{3}} = 0 \quad \text{or} \quad 2 + x^{\frac{1}{3}} = 0$

$\therefore\ x = 0$ or $x = -8$

Substitute $x = 0$ or $x = -8$ into (2)

$\therefore y = -1$ or $y = 7$

Thus $A(0, -1)$ and $B(-8, -7)$.

Length of AB

$= \sqrt{\left(0 - (-8)\right)^2 + \left(-1 - 7\right)^2}$

$= \sqrt{128}$

$= 8\sqrt{2}$

(c) Given that the point D has parameter -1, i.e. $t = -1$.

Substitute $t = -1$ into $x = t^3$, $y = 2t^2 - 1$

$\therefore\ D(-1, 1)$.

Given that the point E has parameter p, i.e. $t = p$

substitute $t = p$ into $x = t^3$, $y = 2t^2 - 1$

$\therefore\ E(p^3, 2p^2 - 1)$

Midpoint DE

$= F\left(\dfrac{p^3 - 1}{2}, \dfrac{(1 + 2p^2) - 1}{2}\right)$

$= \left(\dfrac{p^3 - 1}{2}, p^2\right)$

Let the coordinates of midpoint $DE = (x, y)$, i.e. $(x, y) = \left(\dfrac{p^3 - 1}{2}, p^2\right)$

$\therefore \quad x = \dfrac{p^3 - 1}{2}$ ◃ express x in terms of p and $y = p^2$ .... (4)

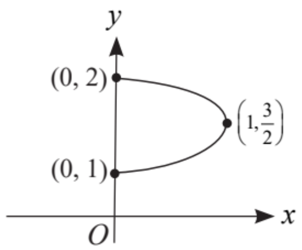
$p = (2x + 1)^{\frac{1}{3}}$ .... (3)

Subsititue (3) into (4) : $y = \left[(2x + 1)^{\frac{1}{3}}\right]^2$

$\therefore$ the cartesian equation of the curve is $y = (2x + 1)^{\frac{2}{3}}$.

22
Solution

(a) The graph of $x=\sin 2t,\ y=\cos^2 t+1,\ 0\le t\le \frac{\pi}{2}$



(a) Given $x=\sin 2t$ $\qquad$ $y=\cos^2 t+1$

$$\frac{dx}{dt}=2\cos 2t \qquad \frac{dy}{dt}=-2\cos t\sin t$$

(b) Using Chain Rule,

$$\frac{dy}{dx}=\frac{-2\cos t\sin t}{2\cos 2t}$$
$$=-\frac{-\sin 2t}{2\cos 2t}$$
$$=-\frac{1}{2}\tan 2t$$

$\therefore\ a=-\frac{1}{2}$ and $b=2$

(c) General equation of tangent to C

$$y-(\cos^2 t+1)=-\frac{1}{2}\tan 2t(x-\sin(2t)) \quad \triangleleft \text{ use } y-y_1=m(x-x_1) \text{ .... (A)}$$

The tangents meets at $\left(2,\frac{3}{2}\right)$.

Substitute $\left(2,\frac{3}{2}\right)$ into (A)

$$\frac{3}{2}-(\cos^2 t+1)=-\frac{1}{2}\tan 2t(2-\sin(2t))$$

From GC,
$t=0.26180$ or 1.30900

Substitute $t=0.26180$ into $x=\sin 2t,\ y=\cos^2 t+1$.
$x=0.500,\ y=1.933$

Substitute $t=1.30900$ into $x=\sin 2t,\ y=\cos^2 t+1$.
$x=0.500,\ y=1.067$

Coordinates of P and Q are $(0.500,\ 1.933)$ and $(0.500,\ 1.067)$.

(d) Given $x = \sin 2t$ .... (1)

and $y = \cos^2 t + 1$

$$y = \frac{1}{2}(\cos 2t + 1) + 1$$

$$2\left(y - \frac{3}{2}\right) = \cos 2t \text{ .... (2)}$$

Using Trigogometric Identities, $\sin^2 2t + \cos^2 2t = 1$ .... (3)
Substituting (1) and (2) into (3).

$$(x)^2 + \left[2\left(y - \frac{3}{2}\right)\right]^2 = 1$$

$\therefore$ the cartesian equation is $x^2 + 4\left(y - \frac{3}{2}\right)^2 = 1,\ \ 0 \le x \le 1.$

23
Solution

(a) Given $x = h + 12\sec t$

$$\sec t = \frac{x-h}{12} \text{ .... (1)}$$

and $y = k + 3\tan t$

$$\tan t = \frac{y-k}{3} \text{ .... (2)}$$

Using Trigogometric Identities, $1 + \tan^2 t = \sec^2 t$ .... (3)
Substituting (1) and (2) into (3).

$$\left(\frac{y-k}{3}\right)^2 + 1 = \left(\frac{x-h}{12}\right)^2$$

$\therefore$ the cartesian equation is $\dfrac{(x-h)^2}{12^2} - \dfrac{(y-k)^2}{3^2} = 1$ (Shown), where $a = 12$, $b = 3$

(b) Find the intersection between the asymptotes

Let $y = \dfrac{x+3}{4}$ .... (1)

and $y = \dfrac{29-x}{4}$ .... (2)

$(1)+(2):$ $2y = 8$

$y = 4$

Substitute $y = 4$ into (1):

$x = 4(4) - 3$

$= 13$

Hence, the two asymptotes intersect at (13, 4).

Since (h, k) is the point of intersection of the asymptotes, $h = 13, k = 4$ (Shown)

Alternative Method

Equation of hyperbola $\dfrac{(x-h)^2}{12^2} - \dfrac{(y-k)^2}{3^2} = 1$.

The equations of the asymptotes are $y - k = \pm\dfrac{1}{4}(x-h)$, i.e. $y = \dfrac{x-h+4k}{4}$ and $y = \dfrac{-x+h+4k}{4}$.

Given that $y = \dfrac{x+3}{4}$ and $y = \dfrac{29-x}{4}$ are asymptotes of C

Compare $y = \dfrac{x-h+4k}{4}$ and $y = \dfrac{x+3}{4}$.

$-h + 4k = 3$ .... (1)

Compare $y = \dfrac{-x+h+4k}{4}$ and $y = \dfrac{-x+29}{4}$.

$h + 4k = 29$ .... (2)

$(1)+(2):$ $8k = 32$

$k = 4$

Substitute $k = 4$ into (1):

$-h + 4(4) = 3$

$h = 13$

(c) When $x = 0$,

$$\frac{(-13)^2}{12^2} - \frac{(y-4)^2}{3^2} = 1$$

$$y = \frac{11}{4} \text{ or } \frac{21}{4}$$

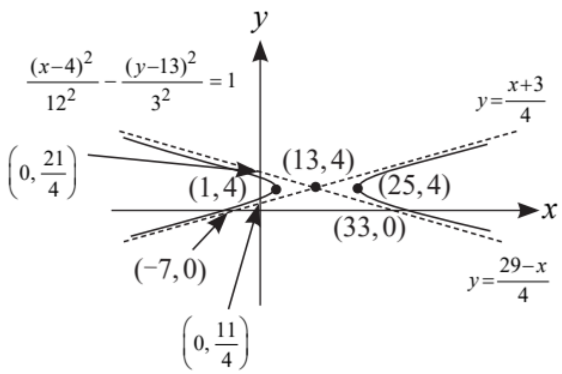
When $y = 0$,

$$\frac{(x-13)^2}{12^2} - \frac{(-4)^2}{3^2} = 1$$

$x = -7$ or 33

centre $= (13,\ 4)$ and vertices are $(1, 4)$ and $(25, 4)$.

The graph of $\dfrac{(x-13)^2}{12^2} - \dfrac{(y-4)^2}{3^2} = 1$.



Exercise 2

D Applications

24
Solution

(a) Let the centre of the circle be (h, k) and the radius be r.

Equation of the circle is $(x-h)^2+(y-k)^2=r^2$

Given that the centre of the circle lies on the y-axis, i.e. $x=0$.

$\therefore\ h=0$.

Substitute the points into the circle equation:

$(2.5)^2+k^2=r^2$ .... (1)

$(3-k)^2=r^2$ .... (2)

Solving (1) and (2) simultaneously.

$k=\dfrac{11}{24}$ and $r=\dfrac{61}{24}$

$\therefore\ x^2+\left(y-\dfrac{11}{24}\right)^2=\left(\dfrac{61}{24}\right)^2$ (Shown) .... (1)

(b) Substituting $x=1.5$ into (1).

$$(1.5-0)^2+\left(y-\frac{11}{24}\right)^2=\left(\frac{61}{24}\right)^2$$

$$(1.5)^2+\left(y-\frac{11}{24}\right)^2=\left(\frac{61}{24}\right)^2$$

$$y-\frac{11}{24}=\pm\sqrt{\left(\frac{61}{24}\right)^2-(1.5)^2}$$

$$y=\frac{11}{24}+\sqrt{\left(\frac{61}{24}\right)^2-1.5^2}\ \text{ or }\ y=\frac{11}{24}-\sqrt{\left(\frac{61}{24}\right)^2-1.5^2}$$

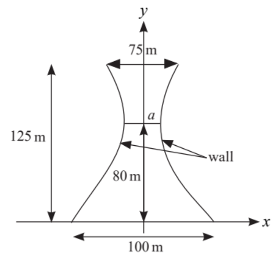
$\therefore\ y=2.51$ m or $y=-1.59$ (rejected since height cannot be - ve)

The maximum allowable height of the bus above the road surface is 2.51 m

25
Solution

(a) Since the narrowest part is at height 80mm, the centre of the hyperbola is at $(0, 80)$.

$\therefore\ k = 80.$



Substitute $(-50, 0)$ into $\dfrac{x^2}{a^2} - \dfrac{(y-80)^2}{b^2} = 1$

$$\frac{(-50)^2}{a^2} - \frac{(-80)^2}{b^2} = 1 \text{ .... (1)}$$

Substitute $(-37.5, 125)$ into $\dfrac{x^2}{a^2} - \dfrac{(y-80)^2}{b^2} = 1$

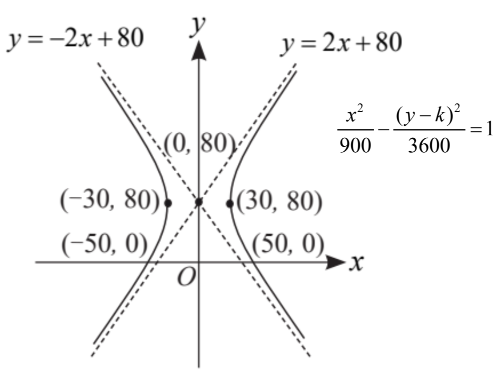
$$\frac{(-37.5)^2}{a^2} - \frac{(45)^2}{b^2} = 1 \text{ .... (2)}$$

Using GC to solve (1) and (2) simultaneously

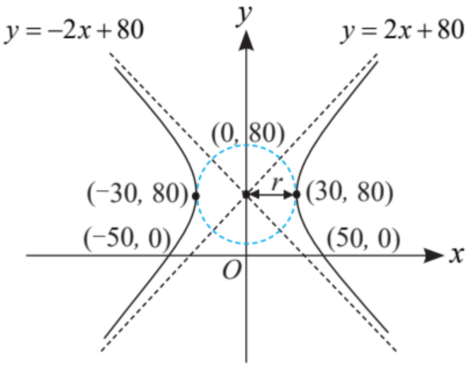
$a^2 = 900$ and $b^2 = 3600$

$\therefore\ k = 80,\ a^2 = 900$ and $b^2 = 3600$

(b)



(c)



The equation $x^2+(y-80)^2=r^2$ represents a circle with centre $(0, 80)$ and radius $= r$.
For C intersects the circle at exactly two distinct points, $r = 30$.

$\therefore\ r = 30$

26
Solution

(a)(i) $a = b = 1$

(ii) Set of values for (a, b) combinations are
(1, 1), (1, 3), (3, 1), (3, 3), (5, 1) and (5, 3).

(b)(i) $\dfrac{(x-3)^2}{3^2} + \dfrac{(y-2)^2}{2^2} = 1$

$(x-1)^2 + (y-1)^2 = 1$

$\downarrow$ Replace x with $\dfrac{x}{3}$

$\left(\dfrac{x}{3}-1\right)^2 + (y-1)^2 = 1$

$\downarrow$ Replace y with $\dfrac{y}{2}$

$\left(\dfrac{x}{3}-1\right)^2 + \left(\dfrac{y}{2}-1\right)^2 = 1$

$\therefore$ the equation of the largest possible elliptical plate is $\left(\dfrac{x}{3}-1\right)^2 + \left(\dfrac{y}{2}-1\right)^2 = 1$.

(ii) Description of sequence of transformations
Scaling parallel to x-axis with a factor of 3.
Scaling parallel to y-axis with a factor of 2.

Exercise 2

E Mixed Exercise

27
Solution

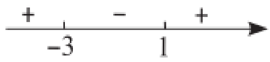
(a) $y = \dfrac{x^2 + x + 1}{x + 1}$

$y(x+1) = x^2 + x + 1$

$x^2 + (1-y)x + (1-y) = 0$

For quadratic equation to have real roots, discriminant ≥ 0.

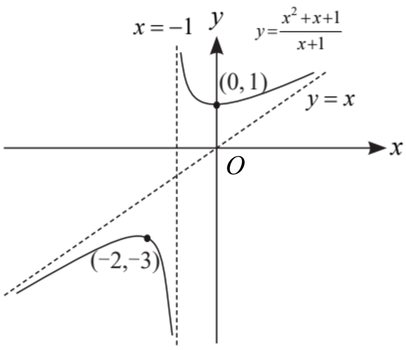
$$(1-y)^2 - 4(1)(1-y) \geq 0$$
$$1 - 2y + y^2 - 4 + 4y \geq 0$$
$$y^2 + 2y - 3 \geq 0$$
$$(y+3)(y-1) \geq 0$$



$\therefore\ y \leq -3$ or $y \geq 1$

Hence, $\mathrm{R_f} = (-\infty, -3] \cup [1, \infty)$

(b)



(c) $\dfrac{(x^2 + x + 1)^2}{(x+1)^2} + (x+2)^2 = k^2$

$$y^2 + (x+2)^2 = k^2$$

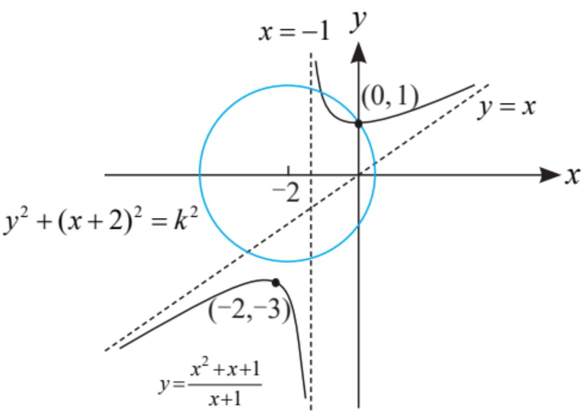
O

Consider diatance from centre of circle to y-intercept of curve

$\sqrt{(0-(-2))^2 + (1-0)^2} = \sqrt{5}$

For the circle to intersect the graph with one positive real root and exactly one negative real root, the radius the circle must be $\sqrt{5} < |k| < 3$.

$\therefore\ \sqrt{5} < |k| < 3$



28
Solution

(a) Using long division, $y = \dfrac{x^2 - x + 7}{x - 2}$

$$= x + 1 + \frac{9}{x-2}.$$

Equations of asymptotes : $x = 2$, $y = x + 1$.

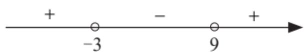
(b) Given $y = \dfrac{x^2 - x + 7}{x - 2}$

$(x-2)y = x^2 - x + 7$

$x^2 + (-1-y)x + (7+2y) = 0$

For quadratic equation has no real roots of x, discriminant < 0.

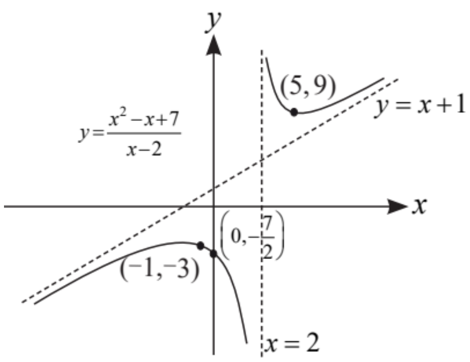
$$(-1-y)^2 - 4(1)(7+2y) < 0$$
$$y - 6y - 27 < 0$$
$$(y+3)(y-9) < 0$$



$\therefore\ -3 < y < 9$

$a = -3$ and $b = 9$.

(c) The graph of $y = \dfrac{x^2 - x + 7}{x - 2}$.



(d) Refer to the graph. For C to be increasing and concave downwards,

$x < -1$

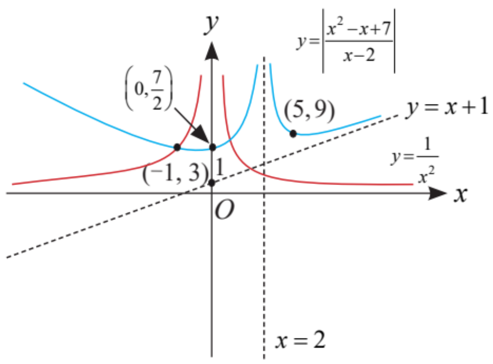
Learning point:
For C to be increasing and concave upwards, then $x > 5$.

(e) $\left|x^2-x+7\right|=\left|\dfrac{2-x}{x^2}\right|$

$\left|\dfrac{x^2-x+7}{2-x}\right|=\left|\dfrac{1}{x^2}\right|$

$\left|\dfrac{x^2-x+7}{x-2}\right|=\dfrac{1}{x^2}$

Add the graph $y=\dfrac{1}{x^2}$ on the same diagram.



Hence, the number of real solutions is 2.

Learning point:

The number of real solutions to the equation $\left|x^2-x+7\right|=\left|\dfrac{2-x}{x^2}\right|$ refers to the number of intersection between the graphs.

29
Solution

(a) Let the denominator of $y = \dfrac{x^2 + ax + b}{c - x}$ be zero.

i.e. $c - x = 0$.

$\therefore\ x = c$ which is the vertical asymptote.

Given $x = -2$ is a vertical asymptote, $\therefore\ c = -2$.

C passes through $(-4, 2)$ and $(0, -6)$.

Substituting $(-4, 2)$ into $y = \dfrac{x^2 + ax + b}{c - x}$.

$$2 = \frac{(-4)^2 + a(-4) + b}{-2 - (-4)}$$

$-4a + b = -12$ .... (1)

Substituting $(0, -6)$ into $y = \dfrac{x^2 + ax + b}{c - x}$.

$$-6 = \frac{(0)^2 + a(0) + b}{-2 - (0)}$$

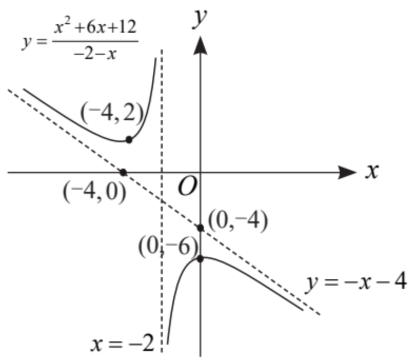
$$b = 12$$

Substituting $b = 12$ into (1)

$$a = 6$$

$\therefore\ a = 6, b = 12$ and $c = -2$

(b) The graph of $y = \dfrac{x^2 + 6x + 12}{-2 - x}$.



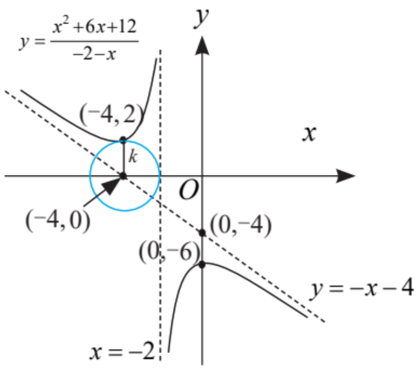
(c) $(x+4)^2+\left(\dfrac{x^2+ax+b}{c-x}\right)^2=k^2$

Replace $y=\dfrac{x^2+ax+b}{c-x}$

$$[x-(-4)]^2+(y)^2=k^2$$

$$(x+4)^2+y^2=k^2 \quad \triangleleft \text{ the equaiton } (x+4)^2+y^2=k^2 \text{ is a circle with radius } k \text{ units, centred at } (-4,\ 0).$$

Add the circle with radius k at centred $(-4, 0)$.



Refer to the diagram. For the equation $(x+4)^2+\left(\dfrac{x^2+ax+b}{c-x}\right)^2=k^2$ has no real roots,

$0<k<2$.

30
Solution

(a) Given $y = \dfrac{1}{x} + \dfrac{2}{x^2}$

$$y = \frac{1}{x} + \frac{2}{x^2}$$
$$yx^2 = x + 2$$
$$yx - x - 2 = 0$$

For the values that y can take, $x \in \mathbb{R}$. Discriminant ≥ 0

$$1 - 4(y)(-2) \geq 0$$
$$y \geq -\frac{1}{8}$$

Solution set $= \left\{ y \in \mathbb{R} : y \geq -\dfrac{1}{8} \right\}$

Alternative Method

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2} - \frac{4}{x^3}$$

At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

i.e. $-\dfrac{1}{x^2} - \dfrac{4}{x^3} = 0$

$\therefore \quad x = -4$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{x^3} + \frac{12}{x^4}.$$

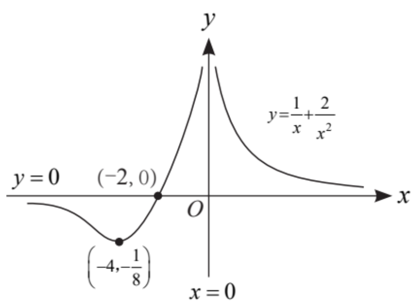
When $x = -4$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{-64} + \frac{12}{256} = \frac{1}{64} > 0$$

$\therefore \left(-4, -\dfrac{1}{8}\right)$ is minimum point

Thus the solution set $= \left\{ y \in \mathbb{R} : y \geq -\dfrac{1}{8} \right\}$

(b) The graph of $y = \dfrac{1}{x} + \dfrac{2}{x^2}$



Asymptotes : $x = 0$, $y = 0$

(c) At $x=-1.5$, substituting into $y=\frac{1}{x}+\frac{2}{x^2}$.

$\therefore\ y=\frac{2}{9}$

At $x=-1$, substituting into $y=\frac{1}{x}+\frac{2}{x^2}$.

$\therefore\ y=1$

At $x=1$, substituting into $y=\frac{1}{x}+\frac{2}{x^2}$.

$\therefore\ y=3$

Substituting $x=-1.5$ and $y=\frac{2}{9}$ into $y=ax^2+bx+c$

$\frac{2}{9}=a(-1.5)^2+b(-1.5)+c$

$2.25a-1.5b+c=\frac{2}{9}$ .... (1)

Substituting $x=-1$ and $y=1$ into $y=ax^2+bx+c$

$1=a(-1)^2+b(-1)+c$

$a-b+c=1$ .... (2)

Substituting $x=1$ and $y=3$ into $y=ax^2+bx+c$

$3=a(1)^2+b(1)+c$

$a+b+c=3$ .... (3)

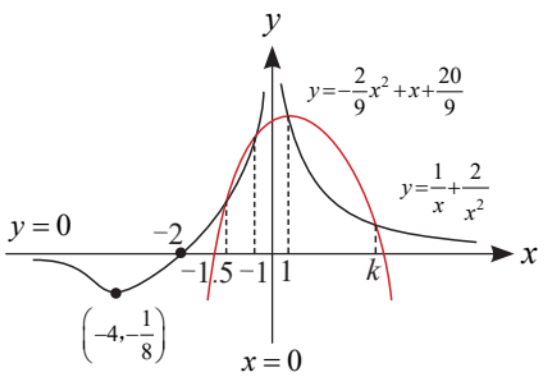
Using GC to solve (1), (2) and (3)

$\therefore \quad a=-\frac{2}{9}, b=1, c=\frac{20}{9}$

Substituting $a=-\frac{2}{9}$, $b=1$ and $c=\frac{20}{9}$ into $y=ax^2+bx+c$.

$\therefore\ y=-\frac{2}{9}x^2+x+\frac{20}{9}$.

Add the graph $y=-\frac{2}{9}x^2+x+\frac{20}{9}$ on the same diagram.



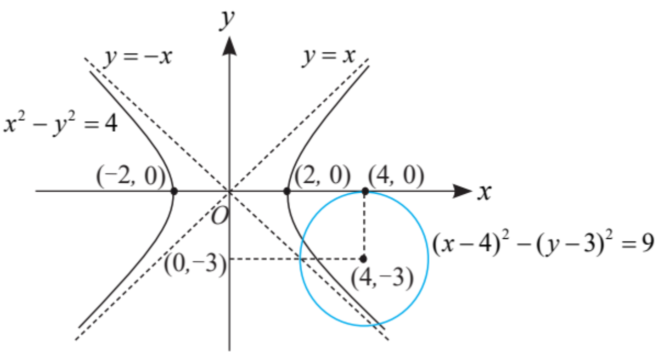
Use GC to find the point of interesctions of $y=-\frac{2}{9}x^2+x+\frac{20}{9}$ and $y=\frac{1}{x}+\frac{2}{x^2}$

$\therefore x=-1.5,\ -1,\ 1$ and 6.

$\therefore\ k=6$

31
Solution

(a)



For C_1 : Circle with centre $(4,-3)$ and radius 3
x - intercept : $(4, 0)$
$(x-4)^2+(y+3)=9$
For C_2 : Hyperbola with centre $(0, 0)$
Asymptotes : $y=x$ and $y=-x$
x - intercepts : $(-2, 0)$ and $(2, 0)$

(b) Substitute $x=3\sin\theta+4$, $y=3\cos\theta-3$ into $x^2-y^2=4$

$(3\sin\theta+4)^2-(3\cos\theta-3)^2=4$

$(9\sin^2\theta+24\sin\theta+16)-(9\cos^2\theta-18\cos\theta+9)=4$

$9(\sin^2\theta-\cos^2\theta)+24\sin\theta+18\cos\theta+3=0$

$3(\sin^2\theta-\cos^2\theta)+8\sin\theta+6\cos\theta+1=0 \,.........(1)$ (Shown)

(c) Solve (1) using G.C.
$\theta=2.5083$ or 5.6014
Substitute $\theta=2.5083$ into $x=3\sin\theta+4$, $y=3\cos\theta-3$
$(5.78,-5.42)$
Substitute $\theta=5.6014$ into $x=3\sin\theta+4$, $y=3\cos\theta-3$
$(2.11,-0.671)$

$\therefore$ the intersection points are $(5.78,-5.42)$ and $(2.11,-0.671)$.

32
Solution

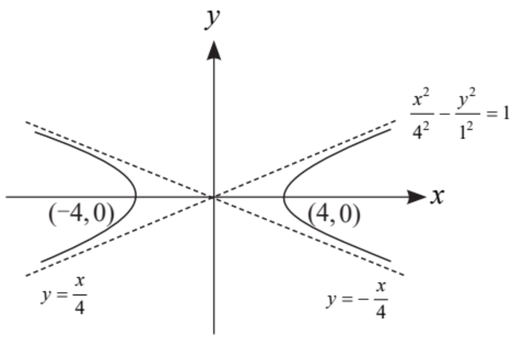
(a) $x^2 - 16y^2 = 16$

$\frac{x^2}{4^2} - \frac{y^2}{1^2} = 1$

Asymptotes : $y = \pm\frac{1}{4}x$

Centre : (0, 0)

Vertices : (4, 0), (–4, 0)



(b) Every line $y = mx$ passes through $(0, 0)$ and must have a steeper gradient compared to the asymptotes of C.

$\therefore$ the range of values of m: $m \geq \frac{1}{4}$ or $m \leq -\frac{1}{4}$

(c) $x - 4 = a\sin\theta$............... (1)

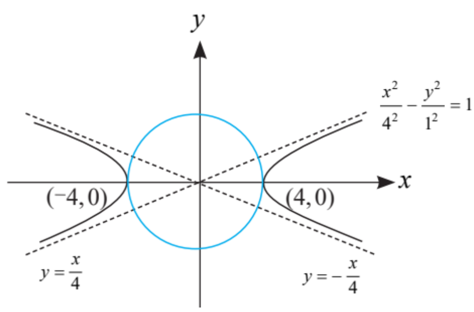
$y = a\cos\theta$.............. (2)

$(1)^2 + (2)^2$:

$a^2\sin^2\theta + a^2\cos^2\theta = (x-4)^2 + y^2$

$\therefore\ (x-4)^2 + y^2 = a^2$

Add the graph $(x-4)^2 + y^2 = a^2$ on the same diagram.



For C_1 to intersect C at four distinct points, the range of a is $a > 8$.

Learning point:

The letter "a" refers to the radius of the circle $(x-4)^2 + y^2 = a^2$.

Exercise 2

F Higher Order Questions

33
Solution

(a) $y = \dfrac{x^2+2}{2x+1}$ ◁ peform long division

$= \dfrac{x}{2} - \dfrac{1}{4} + \dfrac{9}{4(2x+1)}$

Equations of asymptotes

$x = -\dfrac{1}{2}$ is a vertical asymptote.

$y = \dfrac{x}{2} - \dfrac{1}{4}$ is an oblique asymptote.

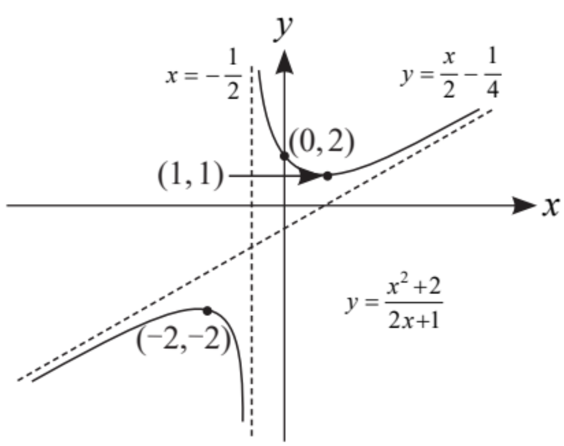
Axial intercept : When $x = 0$, $y = 2$

No intersection with x-axis

Use G.C. to find turning point.

Minimum point $(1, 1)$ and Maximum point $(-2, -2)$

The graph of $y = \dfrac{x^2+2}{2x-1}$.



(b) The oblique asymptote $y=\frac{x}{2}-\frac{1}{4}$ intersects the y-axis at P.

$\therefore$ Substitute $x=0$ into $y=\frac{x}{2}-\frac{1}{4}$

So, $y=-\frac{1}{4}$. $\therefore$ $P\left(0,-\frac{1}{4}\right)$

Substitute $y=0$ into $4y=mx-1$.

$\text{LHS}=4\left(-\frac{1}{4}\right)$

$=-1$

Substitute $x=0$ into $4y=mx-1$.

$\text{RHS}=m(0)-1$

$=-1$

Since LHS = RHS

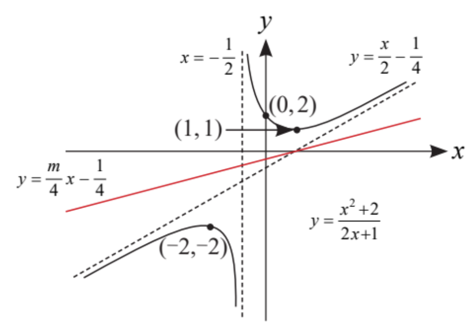
$\therefore$ P lies on $4y=mx-1$ for all real values of m. (Shown)

(c) Given $4x^2+8=(mx-1)(2x+1)$

$$\frac{x^2+2}{2x+1}=\frac{m}{4}x-\frac{1}{4}$$

Given $0<m<2$ ◁ divide 2 to all sides

$$0<\frac{m}{4}<\frac{1}{2}$$



$y=\frac{m}{4}x-\frac{1}{4}$ will not intersect with the graph of $y=\frac{x^2+2}{2x+1}$.

Hence there is no real root for the equation $4x^2+8=(mx-1)(2x+1)$.

(d) Since there are no real roots for the equation, $4x^2+8=(mx-1)(2x+1)$, the two roots are complex roots. Since all the coefficients in the equation are real, the roots must occur in complex conjugate pairs, i.e. $x=a+b\mathrm{i}$ or $x=a-b\mathrm{i}$ where $a,b\in\mathbb{R}$.

34
Solution

By long division,

$$y = \frac{-x^2 + ax}{x - 2b}$$
$$= -x + (a - 2b) + \frac{2b(a - 2b)}{x - 2b}$$

Equation of oblique asymptote : $y = -x + (a - 2b)$

Given that $y = -x + 1$ is an asymptote, by comparing

$$a - 2b = 1$$
$$a = 1 + 2b$$

(a) Substituting $a = 1 + 2b$ into $y = -x + (a - 2b) + \dfrac{2b(a - 2b)}{x - 2b}$.

$$\therefore y = \frac{-x^2 + ax}{x - 2b} = -x + 1 + \frac{2b}{x - 2b}.$$
$$\frac{dy}{dx} = -1 - \frac{2b}{(x - 2b)^2}$$

At stationary, $\dfrac{dy}{dx} = 0.$

$$-1 - \frac{2b}{(x - 2b)^2} = 0$$
$$(x - 2b)^2 = -2b$$
$$x = \pm\sqrt{-2b} + 2b$$

For the curve C to have 2 stationary points, $\sqrt{-2b}$ must exist.

$\therefore\ b < 0$

(b) Equations of asymptotes

$x = 2b$ is a vertical asymptote.

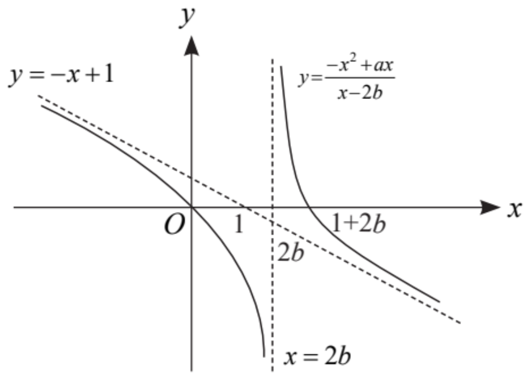
$y = -x + 1$ is an oblique asymptote.

Axial intercept : When $x = 0$, $y = 0$

When $y = 0$, $\dfrac{-x^2 + (1 + 2b)}{x - 2b} = 0$

$$-x^2 + (1 + 2b)x = 0$$
$$x = 0 \quad \text{or} \quad x = 1 + 2b$$

Since $1 < b < 3$, from **(a)**, there must be no stationary points.



(c) Substituting the vertical asymptote $x=2b$ into $y=-x+1$.

$\therefore\ y=-2b+1$

Substituting the vertical asymptote $x=2b$ into $y=m(x-2b)+(1-2b)$.

$\therefore\ y=-2b+1$

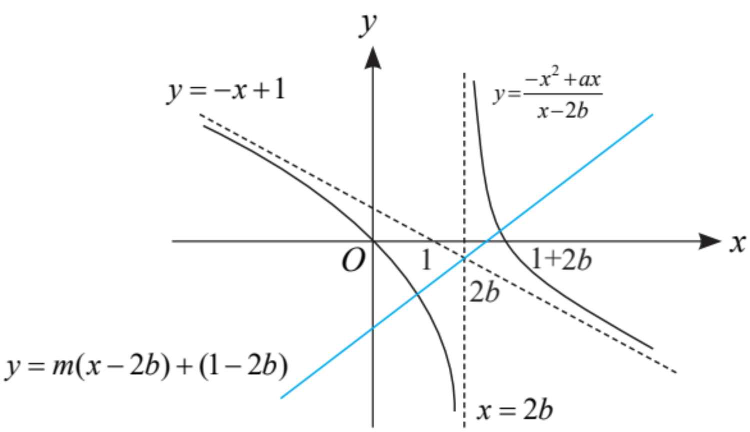
Substituting the oblique asymptote $y=-x+1$ into $y=m(x-2b)+(1-2b)$

$\therefore\ y=-2b+1$

Hence the point $(2b,-2b+1)$ satisfies $y=m(x-2b)+(1-2b)$.

Hence point $(2b,-2b+1)$ on the line $y=m(x-2b)+(1-2b)$.

Add the line $y=m(x-2b)+(1-2b)$ on the the same diagram.



Note that if $m>-1$, the line $y=m(x-2b)+(1-2b)$ cuts C at 2 distinct points.

$\therefore$ the range of values of m is $m>-1$.

35
Solution

(a) $y=\dfrac{x^2-ax+b}{x-2}$

$=x+(2-a)+\dfrac{b+4-2a}{x-2}$ .... (1)

Differentiate (1) with respect to x

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{b+4-2a}{(x-2)^2}$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0.$

i.e. $1-\dfrac{b+4-2a}{(x-2)^2}=0$

$(x-2)^2=b+4-2a$

$x^2-4x+(-b+2a)=0$ .... (2)

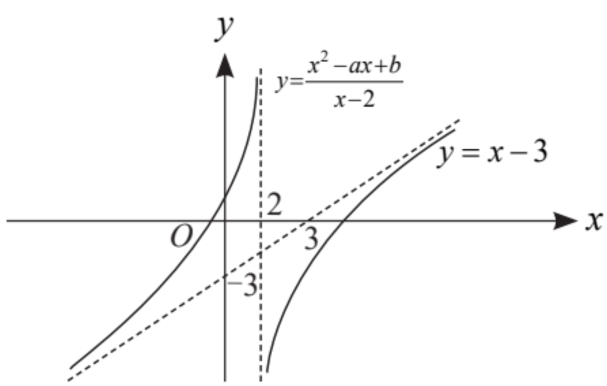
For C to have 2 stationary points, the discriminant of (2) has to be greater than zero, i.e. $b^2-4ac4>0.$

$(-4)^2-4(1)(2a-b)>0$

$b+4-2a>0$

$\therefore\ b>2a-4$ (Shown)

(b) The graph of $y=\dfrac{x^2-ax+b}{x-2}$, where $a=5$ and $-1<b<0.$



Given $2x^2-10x-1=(mx-3)(2x-4)$

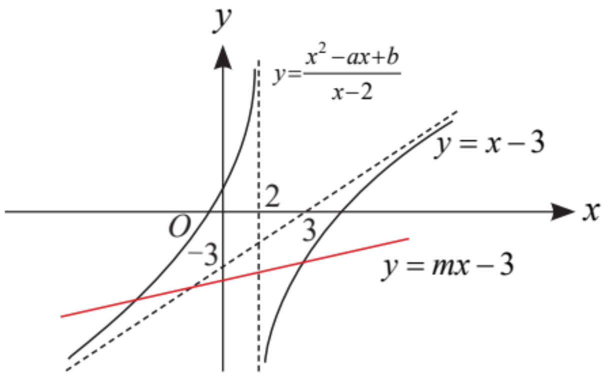
$\dfrac{2x^2-10x-1}{(2x-4)}=mx-3$

$\dfrac{2x^2-10x-1}{(2x-4)}=mx-3$

$\dfrac{x^2-5x-\frac{1}{2}}{x-2}=mx-3,$ where $a=5, b=-\dfrac{1}{2}$

Sketch the line $y=mx-3,$ where $m<1$ on the same diagram as shown.

There are 2 roots for all $m<k.$



Given $\quad 2x^2-10x-1=(mx-3)(2x-4)$

$$\frac{2x^2-10x-1}{(2x-4)}=mx-3$$

$$\frac{x^2-5x-0.5}{(x-2)}=mx-3$$

Note that if $m=1,\ y=mx-3$ is parallel to $y=x-3$.

If $m<1,$ the slope of the line $y=mx-3$ becomes flatter. This line will cut the curve $y=\dfrac{x^2-5x-0.5}{(x-2)}$ at two distinct points.

$\therefore$ the largest possible value of k is 1.

36
Solution

(a) Given $y = \dfrac{6x}{x^2 + ax + 1}$ .... (1)

Consider the denominator of (1)

$x^2 + ax + 1 = \left(x + \dfrac{a}{2}\right)^2 + 1 - \dfrac{a^2}{4}$ $\triangleleft$ completing the square

Given $0 < a < 2$.

$$0 < a^2 < 4 \quad \triangleleft \text{ sqaure both sides}$$

$$0 < \frac{a^2}{4} < 1$$

$$-1 < -\frac{a^2}{4} < 0$$

$$0 < 1 - \frac{a^2}{4} < 1$$

$\therefore \left(x + \dfrac{a}{2}\right)^2 + 1 - \dfrac{a^2}{4} > 0 \ \forall \ x \in \mathbb{R}.$

Since the denominator is always positive,

$\therefore C$ has no vertical asymptote.

Differentiate (1) with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(x^2 + ax + 1\right)6 - 6x\left(2x + a\right)}{\left(x^2 + ax + 1\right)^2}$$

$$= \frac{6\left(1 - x^2\right)}{\left(x^2 + ax + 1\right)^2} \text{ .... (2)}$$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

i.e. $\dfrac{6(1 - x^2)}{\left(x^2 + ax + 1\right)^2} = 0$

$$(1 - x^2) = 0$$

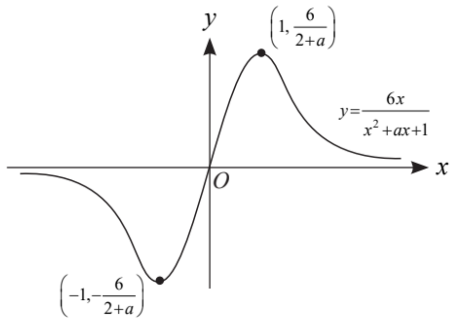
$$x = \pm 1$$

Substituting $x = 1$ into (1): $y = \dfrac{6}{2 + a}$.

Substituting $x = 1$ into (1): $y = -\dfrac{6}{2 - a}$.

$\therefore$ the coordinates of stationary points are $\left(1, \dfrac{6}{2 + a}\right)$ and $\left(-1, -\dfrac{6}{2 - a}\right)$.

The graph of $y = \dfrac{6x}{x^2 + ax + 1}$, where $0 < a < 2$.



(b) If $a = 2$,

$$y = \frac{6x}{x^2 + 2x + 1} = \frac{6x}{(x+1)^2},\ x \neq -1.$$

Asymptotes of C are $x = -1$ and $y = 0$.

At stationary point, $\dfrac{dy}{dx} = 0$.

i.e. $\dfrac{6(1 - x^2)}{(x^2 + ax + 1)^2} = 0$

$$(1 - x^2) = 0$$

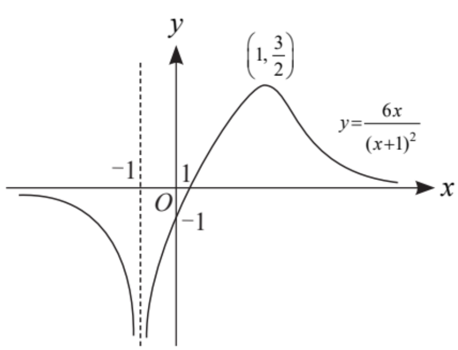
$\therefore$ $x = 1$ or $x = -1$ (Reject, since $x \neq -1$)

Substituting $x = 1$ into $y = \dfrac{6}{2 + a}$.

When $a = 2$,

$\therefore$ the coordinates of stationary points is $\left(1,\ \dfrac{3}{2}\right)$.

The graph of $y = \dfrac{6x}{x^2 + ax + 1}$, where $a = 2$.



Given $6x^2 = (x+1)(x^3 - x)$

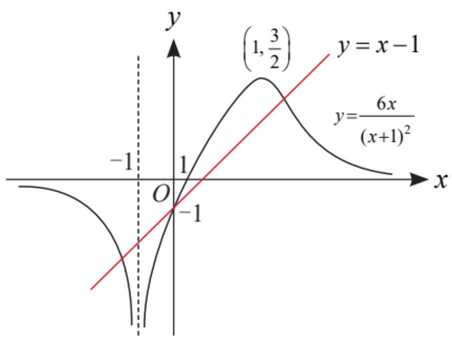
$= x(x+1)^2(x-1)$

$\therefore \quad x\left[6x^2 - x(x+1)^2(x-1)\right] = 0$

$x = 0$ (1 root) or $6x^2 - x(x+1)^2 = 0$

$$\frac{6x}{(x+1)^2} = x - 1$$

Add the line $y = x - 1$ on the same diagram.



From the above diagram, $y = \dfrac{6x}{(x+1)^2}$ and $y = x - 1$ has 3 points of intersection.

Hence, $6x^2 = (x+1)(x^3 - x)$ has 4 real roots.

37
Solution

(a) Given that the line $y = 2x - 1$ is an asymptote of C

$\therefore$ the equation of C can be written as $y = 2x - 1 + \dfrac{k}{x+r}$, $k \in \mathbb{R}$

$$y = 2x - 1 + \frac{k}{x+r}$$
$$= \frac{2x^2 + (2r-1)x + k - r}{x+r} \text{ .... (1)}$$

Given $y = \dfrac{px^2 + qx + 1}{x+r}$ .... (2)

Comparing the coefficient of x^2 with (1) and (2): $p = 2$
Comparing the coefficient of x with (1) and (2): $q = 2r - 1$

$\therefore$ $p = 2$ and $q = 2r - 1$

Alternative Method

$$y = \frac{px^2 + qx + 1}{x+r} \quad \triangleleft \text{ long division}$$
$$= px + q - pr + \frac{1 - qr + pr^2}{x+r}, k \in \mathbb{R} \text{ .... (3)}$$

Comparing (2) and (3):

$p = 2$ and $q - pr = -1$
$q = 2r - 1$

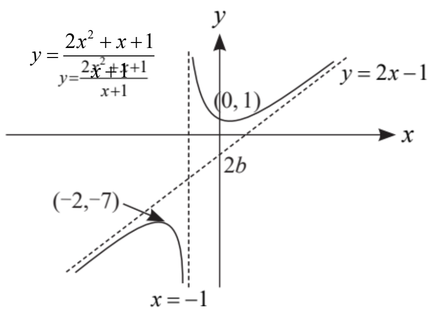
(b) For $r = 1$, $p = 2$, $q = 1$,

$$y = \frac{2x^2 + x + 1}{x+1}$$
$$= 2x - 1 + \frac{2}{x+1}.$$

Equations of asymptotes
$x = -1$ is a vertical asymptote.
$y = 2x - 1$ is an oblique asymptote.
Axial intercept : When $x = 0$, $y = 1$
No intersection with x-axis

Use G.C. to find turning point.
Minimum point $(0, 1)$

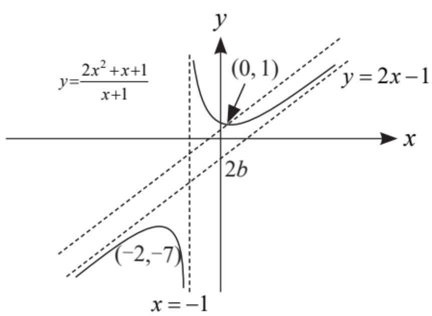
The graph of $y = \dfrac{2x^2 + x + 1}{x + 1}$.



(c) $2x^2 + x + 1 = kx^2 + (k + 1)x + 1$

$2x^2 + x + 1 = (kx + 1)(x + 1)$

$\dfrac{2x^2 + x + 1}{x + 1} = kx + 1$



The line $y = kx + 1$ passes through the point $(0, 1)$, which is the minimum point of the curve.
The oblique asymptote has a gradient of 2. For the line intersects the curve at 2 points, k is $\mathbb{R} \setminus \{0, 2\}$.

$\therefore \ \{k \in \mathbb{R}, k \neq 0, 2\}$

38
Solution

(a) The oblique asymptote that intersects the axes at $(-a,\ 0)$ and $(0,-a)$.

Using these points to form equation of oblique asymptote :

$$y-(-a)=\frac{0-(-a)}{-a-0}[x-0]$$

$$y=-x-a$$

$\therefore$ the equation of oblique asymptote is $y=-x+a$.

Express the equation of the curve as

$y=-x-a+\dfrac{k}{x+1}$, where $k\in\mathbb{R}_0$ .... (1)

The graph passes through $(0,-a-1)$. Substituting the point into (1).

$$-a-1=0-a+k$$

$$k=-1$$

$\therefore$ the equation of the curve is $y=-x-a-\dfrac{1}{x-1}$.

(b) Differentate (1) with respect to x.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-1-\frac{1}{x+1}$$

At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$

i.e. $-1-\dfrac{1}{x+1}=0$

$\therefore\ x=-2$ or $x=0$

Substituting $x=-2$ into equation of the curve $y=-x-a-\dfrac{1}{x-1}$.

$y=-a+3$ and

Substituting $x=0$ into equation of the curve $y=-x-a-\dfrac{1}{x-1}$.

$y=-a-1$

$\therefore$ the coordinates of turning points are $(-2,-a+3)$ and $(0,-a-1)$.

The set values of y for which there are no turning points on the curve is $-a-1<y<-a+3$.

39
Solution

(a)

$$\mathrm{f}(x)=\frac{ax^2+bx+c+2}{x+d} \text{ .... (1)}$$

Since $x=0$ is a vertical asymptote, $\therefore\ d=0$.

$\therefore\ y=ax+b+\dfrac{c+2}{x}$

$y=ax+b$ is the oblique asymptote.

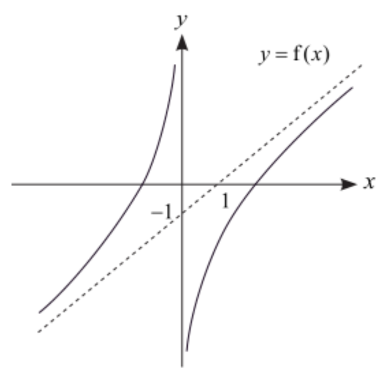
$\therefore\ b=-1$ and $a=1$.

$\therefore\ a=-1,\ b=-1$ and $d=0$.

(b) Substitute $a,\ b$ and d into (1).

$y=x-1+\dfrac{c+2}{x}$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{c+2}{x^2}$



From the diagram, the graph of $y=\mathrm{f}(x)$ is increasing for all $x, x\neq 0$.

$\therefore\ \dfrac{\mathrm{d}y}{\mathrm{d}x}>0$.

i.e. $1-\dfrac{c+2}{x^2}>0$ for all real values of $x,\ x\neq 0$

For $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$, then $c+2<0$.

$\therefore\ c<-2$ (Shown)

(c) If $c=-1$,

$y=x-1+\dfrac{1}{x}$ $\triangleleft$ substituting the values of $a,\ b$ and d

The asymptotes of the curve $y=x-1+\dfrac{1}{x}$

$x=0$ is a vertical asymptote.

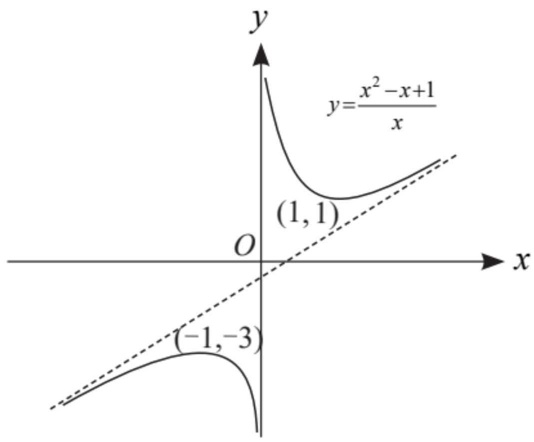
$y=x-1$ is an oblique asymptote.

Axial intercept : The graph has no axial intercepts.

Use G.C. to find turning point.

Minimum point $(1,1)$ and Maximum point $(-1,-3)$

The graph of $y=\dfrac{x^2-x+1}{x}$.

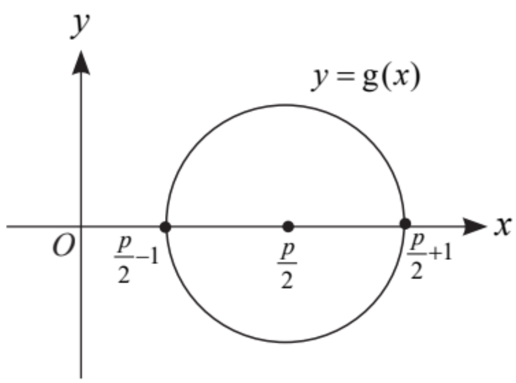


(d) $4x^2+4y^2-4px-4+p^2=0$

$x^2-px+y^2+\dfrac{p^2}{4}-1=0$

$\left(x-\dfrac{p}{2}\right)^2+y^2=1$

g(x) is a circle with centre at $\left(\dfrac{p}{2},\ 0\right)$ and radius 1.



(e) f(x) and g(x) do not intersect if $0<\dfrac{p}{2}<1$ or $\dfrac{p}{2}>1$

$\therefore\ 0<p<2 \quad \text{or} \quad p>2$

40
Solution

(a) $y=\dfrac{(x-1)(bx+1)}{x-2}$

$=bx+b+1+\dfrac{2b+1}{x-2}$

As $x\to\infty$, $\dfrac{2b+1}{x-2}\to 0^{+}$. $y\to(bx+b+1)^{+}$.

As $x\to-\infty$, $\dfrac{2b+1}{x-2}\to 0^{-}$. $y\to(bx+b+1)^{-}$.

The oblique asymptote is $y=bx+b+1$. (Shown)

The vertical asymptote is $x=2$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=b-\dfrac{2b+1}{(x-2)^2}$

At turning point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$.

i.e. $b-\dfrac{2b+1}{(x-2)^2}=0$

$\dfrac{2b+1}{(x-2)^2}=b$

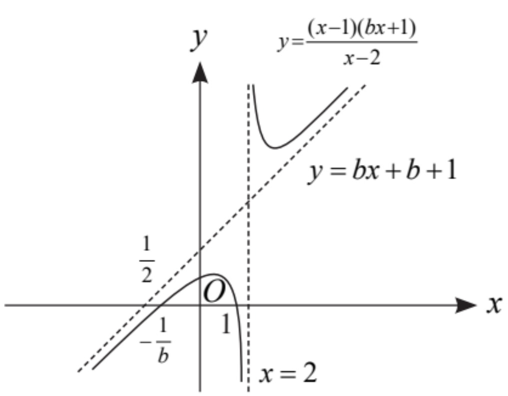
$x=\pm\sqrt{\dfrac{2b+1}{b}}+2$

For C has two stationary points, $\dfrac{2b+1}{b}>0$.

Thus $b(2b+1)>0$. (Shown)

(c)(i) The graph of $y=bx+b+1+\dfrac{2b+1}{x-2}$, where $b>2$.

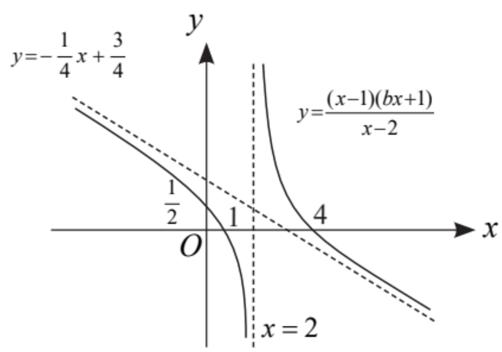
Note that $b>2$, C has two turning points.



(ii) The graph of $y = bx + b + 1 + \dfrac{2b+1}{x-2}$, where $b = -\dfrac{1}{4}$.

Note that $b = -\dfrac{1}{4}$, C has no turning points

The oblique asymptote is $y = -\dfrac{1}{4}x + \dfrac{3}{4}$.

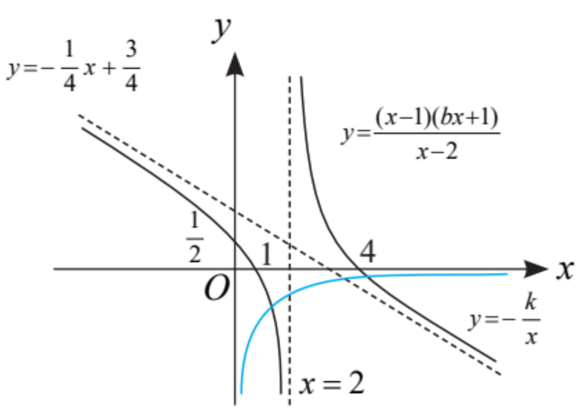


(d) Rearranging $b|x|^3 + (1-b)x^2 + (k-1)|x| - 2k = 0$

$\therefore \quad \dfrac{b|x|^2 + (1-b)|x| - 1}{|x| - 2} = \dfrac{k}{|x|}$

Consider the number of intersection points between C and the curve $y = -\dfrac{k}{x}$ for $x > 0$.

Add the curve $y = -\dfrac{k}{x}$ to the same diagram.



From the sketch, there are 2 intersection points.

Thus the equation $b|x|^3 + (1-b)x^2 + (k-1)|x| - 2k = 0$ has 4 distinct real roots.

41
Solution

(a)(i)

Given $y=\dfrac{ax^2+bx+c}{x-1}$ .... (1)

Substitute $\left(3,\dfrac{23}{2}\right)$ into (1)

$$\frac{a(3)^2+3b+c}{3-1}=\frac{23}{2}$$
$$9a+3b+c=23 \text{ .... (2)}$$

Substitute $(2,10)$ into (1)

$$\frac{a(2)^2+2b+c}{2-1}=10$$
$$4a+2b+c=10 \text{ .... (3)}$$

Differentiate (1) with respect to x.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2ax+b)(x-1)-(ax^2+bx+c)(1)}{(x-1)^2}$$
$$=\frac{ax^2-2ax-b-c}{(x-1)^2}$$

Given $(2,10)$ is a minimum point, i.e. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=2$.

So $0=4a-4a-b-c$

$b+c=0$ .... (3)

Solving (1), (2) and (3) using the GC,

$\therefore\ a=3,\ b=-2, c=2.$

(a)(ii) $y=\dfrac{3x^2-2x+2}{x-1}$

Performing long division,

$$y=3x+1+\frac{3}{x-1}$$

Equations of asymptotes

$x=1$ is a vertical asymptote.

$y=3x+1$ is an oblique asymptote.

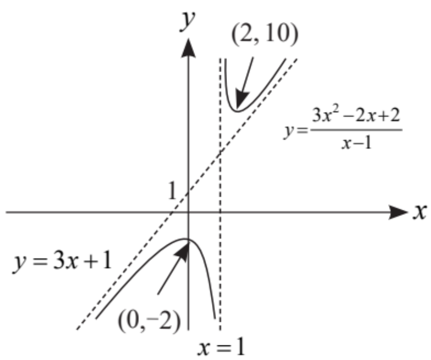
Axial intercept : When $x=0, y=2$

No intersection with x-axis

Use G.C. to find turning point.

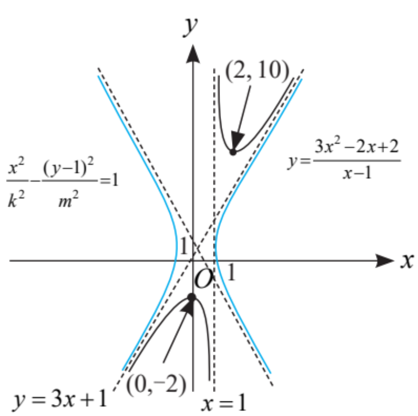
Minimum point $(2,10)$ and Maximum point $(0,-2)$

The graph of $y=\dfrac{3x^2-2x+2}{x-1}$.



(a)(iii)

The equation $\dfrac{x^2}{k^2}-\dfrac{(y-1)^2}{m^2}=1$ represents a hyperbola with equations of asymptotes $y=1\pm\dfrac{m}{k}x.$



Refer to the diagram above. By comparing both asymptotes, observe that the positve gradient of the asymptote of the hyperbola must be at most 3 if the two graphs are not to intersect.

Hence, the set of values of $\dfrac{m}{k}$ is $\left\{\dfrac{m}{k}\in\mathbb{R}:0<\dfrac{m}{k}\leq 3\right\}.$

(b) When $a = b = -1$,

$$y = \frac{-x^2 - x + c}{x-1} \qquad \triangleleft \text{ substituting } a = b = -1 \text{ into } y = \frac{ax^2 + bx + c}{x-1}$$

If $c = 2$,

$$y = \frac{-x^2 - x + 2}{x-1}$$

$$y = -\frac{(x-1)(x+2)}{x-1}$$

$$y = -x - 2,\ x \neq 1$$

C has no stationary point when $c = 2$.

From **(a)**, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{ax^2 - 2ax - b - c}{(x-1)^2}$

When $a = b = -1$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x^2 + 2x + 1 - c}{(x-1)^2} \qquad \triangleleft \text{ substituting } a = b = -1$$

At stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

i.e. $-x^2 + 2x + 1 - c = 0$.

For there to be no stationary points, the equation $-x^2 + 2x + 1 - c = 0$ must have no real solutions for x.

Discriminant < 0

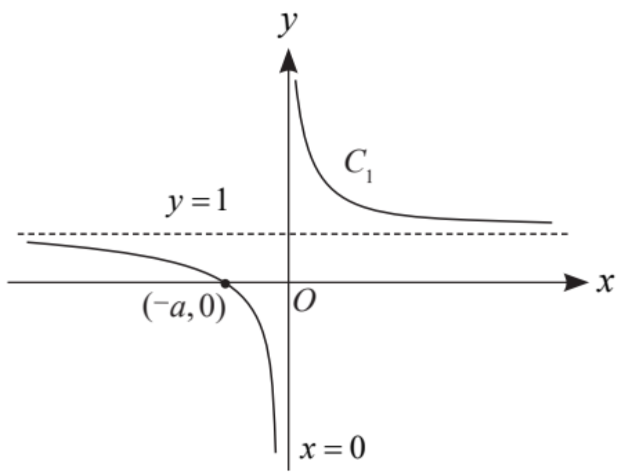
$2^2 - 4(-1)(1-c) < 0$.

$4 + 4(1-c) < 0$

$c > 2$

$\therefore$ the set of values of c is $\{c \in \mathbb{R} : c \geq 2\}$.

42.
Solution



(a) Given $x = \dfrac{a}{t}$ .... (1)

and $y = 1 + t$

$t = y - 1$ .... (2)

Substitute (1) into (2): $x = \dfrac{a}{y-1}$

Substitute $x = \dfrac{a}{y-1}$ into $y = \sqrt{1 + \dfrac{x^2}{a^2}}$

$$\therefore \quad y = \sqrt{1 + \dfrac{\left(\dfrac{a}{y-1}\right)^2}{a^2}} \quad \triangleleft \text{ square both sides}$$

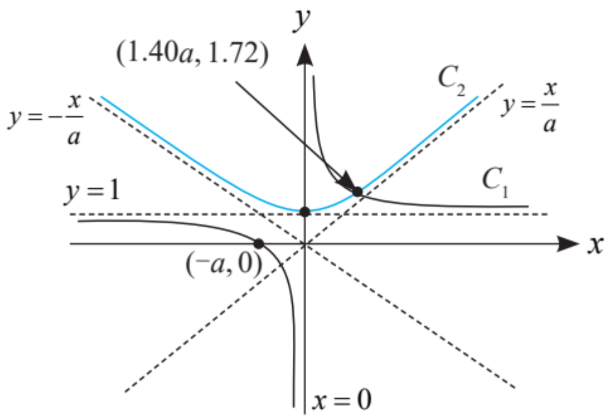
$$y^2 = 1 + \frac{1}{(y-1)^2}$$

$$y^2 - 1 = \frac{1}{(y-1)^2}$$

$$(y^2 - 1)(y-1)^2 = 1$$

$$(y^2 - 1)(y-1)^2 - 1 = 0 \quad \text{(Shown)}$$

(b)



(c) Using GC to solve, $(y-1)^2(y^2-1)-1=0$.

$y=-1.106919$ (rejected) or 1.716673

Substituting $y=1.716673$ into $x=\dfrac{a}{y-1}$.

$x=1.39534a$

$\therefore$ the point of intersection is $(1.40a, 1.72)$.

43
Solution

(a) Given $y = 2x + 1 + \dfrac{b}{x+a}, x \neq -a$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 - \dfrac{b}{(x+a)^2}$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

i.e. $\quad 2 - \dfrac{b}{(x+a)^2} = 0$

$2(x+a)^2 - b = 0$

$2x^2 + 4ax + 2a^2 - b = 0$

Given that C_1 has two stationary points, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ has solution

Discriminant

$= 16a^2 - 4(2)(2a^2 - b)$

$= 8b \geq 0$

$\therefore \quad b > 0$ and $a \in \mathbb{R}$.

Learning point:

If $b = 0,$ then $y = 2x + 1 + \dfrac{b}{x+a}$ becomes a horizontal line.

(b) Given that $a = b = 1,$

$\therefore \; y = 2x + 1 + \dfrac{1}{x+1}$

Equations of asymptotes

$x = -1$ is a vertical asymptote.

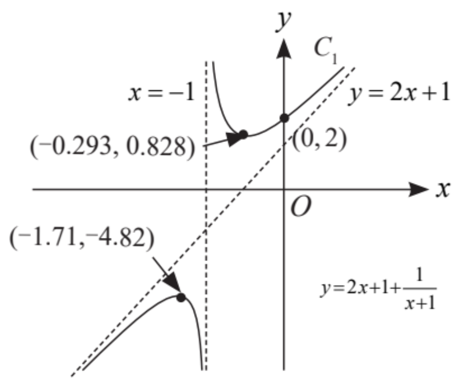
$y = 2x + 1$ is an oblique asymptote.

Axial intercept : No intersection with axes.

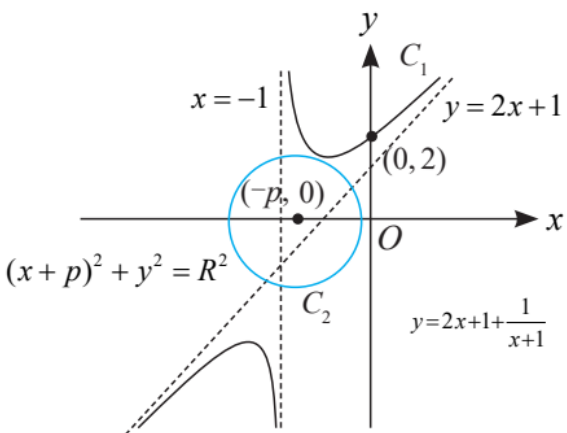
Use G.C. to find turning point.

Minimum point $(1, 1)$ and Maximum point $(-2, -2)$.

The graph of $y = 2x + 1 + \dfrac{1}{x+1}$.



(c) The equation $(x+p)^2 + y^2 = R^2$ is a circle centre $(-p, 0)$ and radius R.
Add the circle on the same diagram.



(d) Given that $a = b = 1$, $\therefore\ \dfrac{dy}{dx} = 2 - \dfrac{1}{(x+1)^2}$.

At stationary points of C_1, $\dfrac{dy}{dx} = 0$.

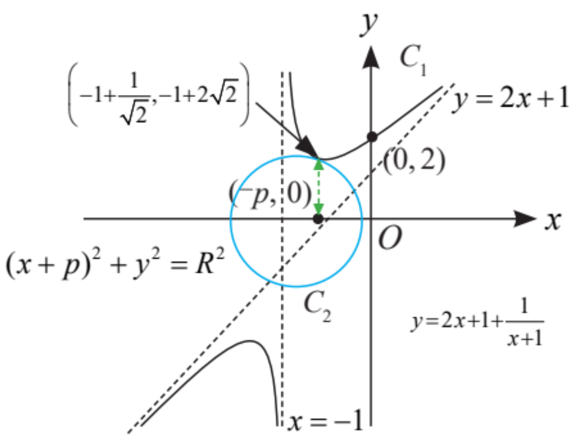
$$(x+1)^2 = \frac{1}{2}$$

$$x = -1 \pm \frac{1}{\sqrt{2}}$$

Substitute $x = -1 + \dfrac{1}{\sqrt{2}}$ into $y = 2x + 1 + \dfrac{1}{x+1}$

$$y = 2\left(-1 + \frac{1}{\sqrt{2}}\right) + 1 + \frac{1}{\frac{1}{\sqrt{2}}}$$

$$= -1 + 2\sqrt{2}$$



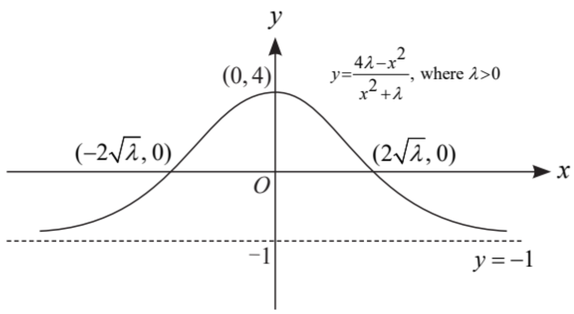
The minimum R for C_1 and C_2 to intersect occurs when the centre of the circle is directly below the turning point $\left(-1 + \dfrac{1}{\sqrt{2}}, -1 + 2\sqrt{2}\right)$.

Thus min $R = 2\sqrt{2} - 1$

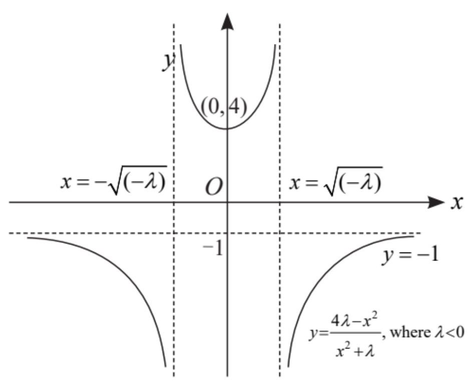
Corresponding value of $p = -\left(-1 + \dfrac{1}{\sqrt{2}}\right) = 1 - \dfrac{1}{\sqrt{2}}$.

44
Solution

(a) The graph of $y=\dfrac{4\lambda-x^2}{x^2+\lambda}$, where $\lambda>0$.



(b) The graph of $y=\dfrac{4\lambda-x^2}{x^2+\lambda}$, where $\lambda<0$.



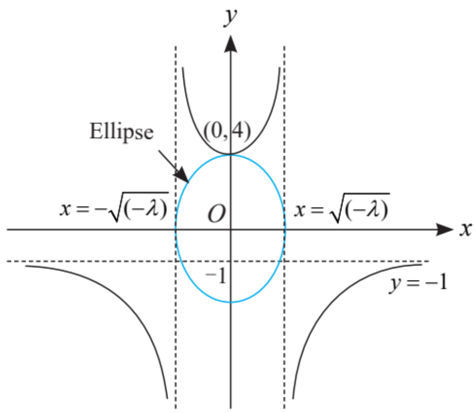
Given $\left(\dfrac{4\lambda-x^2}{kx^2+h\lambda}\right)^2=1+\dfrac{x^2}{\lambda}$

$$-\frac{x^2}{\lambda}+\left(\frac{4\lambda-x^2}{kx^2+h\lambda}\right)^2=1$$

$$\frac{x^2}{(-\sqrt{\lambda})^2}+\frac{1}{h^2}\left(\frac{4\lambda-x^2}{x^2+\lambda}\right)^2=1 \qquad \vartriangleleft\ y=\frac{4\lambda-x^2}{x^2+\lambda}$$

$$\frac{x^2}{(-\sqrt{\lambda})^2}+\frac{1}{h^2}(y)^2=1$$

The equation $\dfrac{x^2}{(-\sqrt{\lambda})^2}+\dfrac{y^2}{h^2}=1$ is an ellipse with centre O.



For one real root, the ellipse must meet the graph in **(b)** at exactly one point.

Hence $h = 4$ and the value of the real root when $h = 4$ is $x = 0$.

45
Solution

(a) Given $y = \dfrac{-4x^2 + 8kx - 5k^2 + 4}{x - k}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-k)(-8x+8k) - (-4x^2 + 8kx - 5k^2 + 4)(1)}{(x-k)^2}$$

$$= \frac{-8x^2 + 8kx + 8kx - 8k^2 + 4x^2 - 8kx + 5k^2 - 4}{(x-k)^2}$$

$$= \frac{-4x^2 + 8kx - 3k^2 - 4}{(x-k)^2}$$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

i.e. $\dfrac{-4x^2 + 8kx - 3k^2 - 4}{(x-k)^2} = 0.$

$-4x^2 + 8kx - 3k^2 - 4 = 0$

For C has two stationary points, the discrimiant of (1) must be positive.

i.e. D > 0

$(8k)^2 - 4(-4)(-3k^2 - 4) > 0$

$64k^2 - 48k^2 - 64 > 0$

$16k^2 - 64 > 0$

$16(k+2)(k-2) > 0$

Therefore, $k < -2$ or $k > 2$.

(b) Given $y = \dfrac{-4x^2 + 8kx - 5k^2 + 4}{x - k}$

Performing long division:

$$\begin{array}{r@{}l} & -4x + 4k \\ x - k \big) & -4x^2 + 8kx + (4 - 5k^2) \\ -) & -4x^2 + 4kx \\ \hline & 4kx + (4 - 5k^2) \\ -) & 4kx - 4k^2 \\ \hline & 4 - k^2 \end{array}$$

$$\therefore\ y = \frac{-4x^2 + 8kx - 5k^2 + 4}{x - k}$$

$$= -4x + 4k - \frac{k^2 - 4}{x - k}$$

The oblique asymptote is $y = -4x + 4k$.

The oblique asymptote cuts the y-axis at (0, 4).

Substituting $x = 0$ and $y = 4$ into $y = -4x + 4k$.

$4k = 4$

$\therefore\ k = 1$

(c) $y = \dfrac{-4x^2+8x-1}{x-1}$

$= -4x+4+\dfrac{3}{x-1}$

Equations of asymptotes

$x=1$ is a vertical asymptote.

$y=-4x+4$ is an oblique asymptote.

Intercepts :

When $x=0$, $y=\dfrac{-1}{-1}=1 \;\therefore (0,1)$

When $y=0$,

$-4x^2+8x-1=0$

$4x^2-8x+1=0$

$x=\dfrac{8\pm\sqrt{64-4(4)(1)}}{2(4)}$

$=\dfrac{8\pm\sqrt{48}}{8}$

$=1\pm\dfrac{\sqrt{16\times3}}{8}$

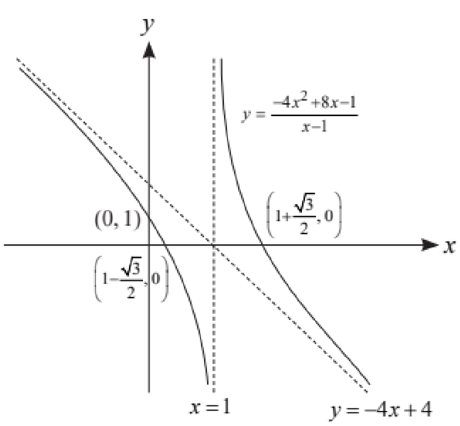
$=1\pm\dfrac{4\sqrt{3}}{8}$

$=1\pm\dfrac{\sqrt{3}}{2}$

$\left(1+\dfrac{\sqrt{3}}{2},0\right)$ or $\left(1-\dfrac{\sqrt{3}}{2},0\right)$

Turning point : From **(a)**, when $k<-2$ or $k>2$, curve C has no turning point.
Since $k=1$, there is no turning points for the curve C.

The graph of $y=\dfrac{-4x^2+8x+1}{x-1}$.



(d) Given $x = 1 + \tan t$,

$x - 1 = \tan t$ .... (1)

and $\quad y = b\sec t$ .... (2)

Using the trigonometric identity : $\tan^2 t + 1 = \sec^2 t$ .... (3)
Substituting (1) and (2) into (3).

$$(x-1)^2 + 1 = \left(\frac{y}{b}\right)^2$$

$$\left(\frac{y}{b}\right)^2 - (x-1)^2 = 1$$

$$y^2 - b^2(x-1)^2 = b^2$$

The cartesian equation of the curve C_1 is $y^2 - b^2(x-1)^2 = b^2$.

(e) The equation $\left(\frac{y}{b}\right)^2 - (x-1)^2 = 1$ represents a hyperbola.

The asymptotes of the hyperbola are $y = \pm b(x-1)$ and centre is (1, 0).
Hence, for the hyperbola to intersect the curve C at most twice, $b \geq 4$.

$\therefore$ the range of value of b is $b \geq 4$.

Learning point:
In order C and C_1 to have at most two intersection points, positive gradient of the hyperbola asymptote must be equal or more than 4. Therefore, $b \geq 4$.

Exercise 2

J Exam Style Questions

46
Solution

(a) Given $y = \dfrac{2x^2 - 3x}{2 - x}$

$2x^2 - 3x = k(2 - x)$

$2x^2 - 3x + kx - 2k = 0$

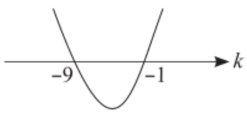
For the values of y to be taken, there are real roots.

Discriminant ≥ 0

$(k-3)^2 - 4(2)(-2k) \geq 0$

$k^2 - 6k + 9 + 16k \geq 0$

$k^2 + 10k + 9 \geq 0$



$(k+1)(k+9) \geq 0$

$\therefore \quad k \leq -9$ or $k \geq -1$

$\therefore$ the range of values of y is $y \leq -9$ or $y \geq -1$.

(b) $y = \dfrac{2x^2 - 3x}{2 - x}$

$= -2x - 1 + \dfrac{2}{2 - x}$

Equations of asymptotes

$x = 2$ is a vertical asymptote.

$y = -2x - 1$ is an oblique asymptote.

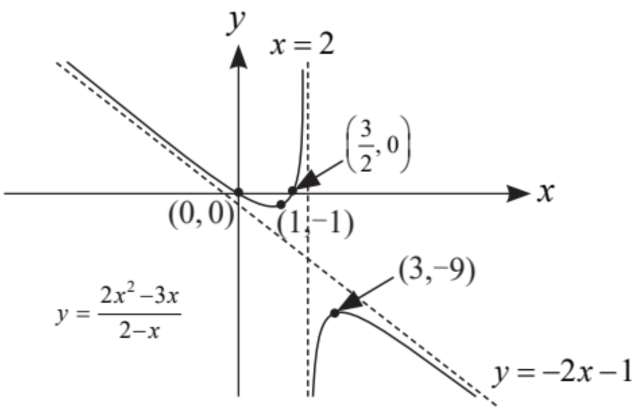
Axial intercept : When $x = 0$, $y = 0$

When $y = 0$, $y = 1.5$

Using G.C. to find turning point.

Minimum point $(1, -1)$ and maximum point $(3, -9)$

The graph of $y = \dfrac{2x^2 - 3x}{2 - x}$.



(c) Given $4x^2(2x-3)^2=(2-x)^2\left[64-(x-1)^2\right]$

$$4(2x^2-3x)^2=(2-x)^2\left[64-(x-1)^2\right]$$

$$4\left[\frac{(2x^2-3x)^2}{(2-x)^2}\right]=64-(x-1)^2$$

$$4\left[\frac{(2x^2-3x)}{(2-x)}\right]^2=64-(x-1)^2$$

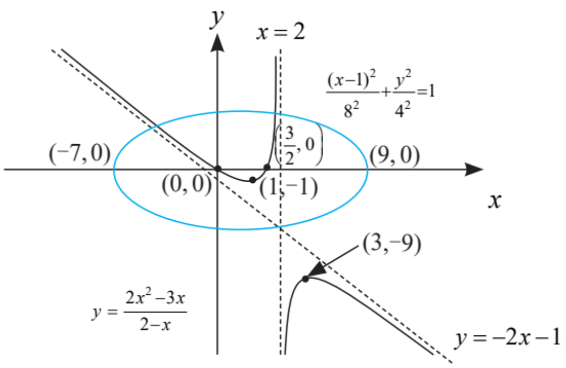
Replace $\frac{(2x^2-3x)}{(2-x)}$ by y.

$$4y^2+(x-1)^2=64$$

$$\frac{y^2}{16}+\frac{(x-1)^2}{64}=1$$

The equation $\frac{(x-1)^2}{8^2}+\frac{y^2}{4^2}=1$ is an ellipse with centre (1, 0). It has semi-major axis 8 units and semi-minor axis 4 units.

Add the graph $\frac{(x-1)^2}{8^2}+\frac{y^2}{4^2}=1$.



From the diagram, the graphs intersect twice.

$\therefore$ there are 2 real roots for the equation $4x^2(2x-3)^2=(2-x)^2\left[64-(x-1)^2\right]$.

47
Solution

(a) $y = \dfrac{2x^2 - 5ax + 2a^2 + 1}{x - a}$

$= 2x - 3a + \dfrac{1 - a^2}{x - a}$

Asymptotes are $y = 2x - 3a$ and $x = a$.

(b) $y = 2x - 3a + \dfrac{1 - a^2}{x - a}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 - \dfrac{1 - a^2}{(x - a)^2}$

At stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\therefore\ 2 - \dfrac{1 - a^2}{(x - a)^2} = 0$

$2(x - a)^2 = 1 - a^2$

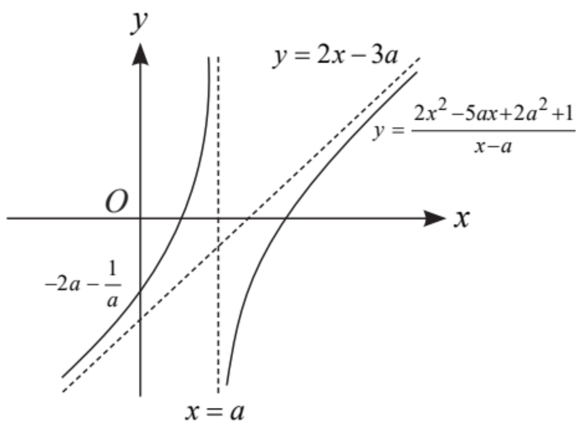
Given that the curve has no stationary points,

$1 - a^2 < 0$

$(1 - a)(1 + a) < 0$

$\therefore\ a < -1$ or $a > 1$

(c) The graph of $y = \dfrac{2x^2 - 5ax + 2a^2 + 1}{x - a}$, for $a > 1$



48
Solution

(a) $d = 4$

Given that $y = 3x + 2$ is an asymptote of of the curve,

we can express the equation of curve as $y = 3x + 2 + \dfrac{k}{x+4}$, where k is a constant.

Equating the equation of the curve

$$3x + 2 + \frac{k}{x+4} = \frac{3x^2 + bx + c}{x+4}$$

Multiplying $x + 4$ throughout,

$$3x^2 + 14x + 8 + k = 3x^2 + bx + c$$

By comparing coefficient of x, $b = 14$.

$\therefore\ b = 14$

(b) $$y = \frac{3x^2 + 14x + c}{x+4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 + 24x + (56 - c)}{(x+4)^2}$$

At the stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

i.e. $\quad 3x^2 + 24x + (56 - c) = 0$

When $x = -2$,

$$12 - 48 + (56 - c) = 0$$
$$c = 20$$

$\therefore$ the value of c is 20.

$$3x^2 + 24x + 36 = 0$$

Using GC, $x = -2$ or $x = -6$

Substituting $x = -2$ into $y = \dfrac{3x^2 + 14x + 20}{x+4}$

$\therefore\ y = 2$

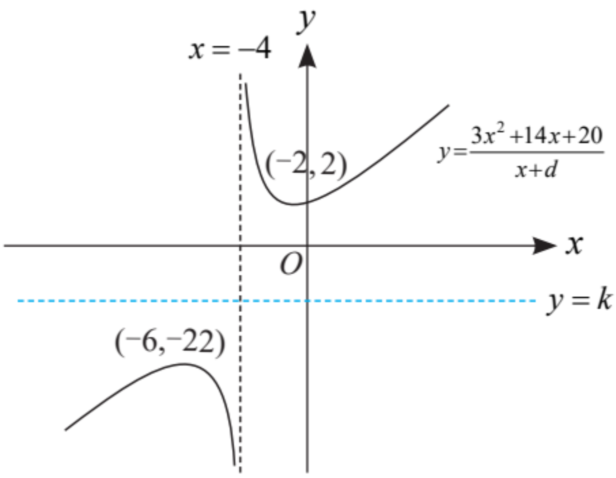
Substituting $x = -6$ into $y = \dfrac{3x^2 + 14x + 20}{x+4}$

$\therefore\ y = -22$

The coordinates of the stationary points are $(-6, -22)$ and $(-2, 2)$

Given $3x^2 + bx + c = k(x+d)$

$$\frac{3x^2 + bx + c}{x+d} = k$$



For the equation $3x^2 + bx + c = k(x+d)$ has no real roots, the line $y = k$ does not intersect the curve.

$\therefore$ the range of value of k is $-22 < k < 2$.

(c) Solve $x = 4$ and $y = 3x + 2$ simultaneously.

$\therefore\ y = -10$

The coordinates where the two 2 asymptotes intersect is $(4, -10)$

Substituting $x = -4$ in $y = mx + 4m - 10$.

$y = m(-4) + 4m - 10$

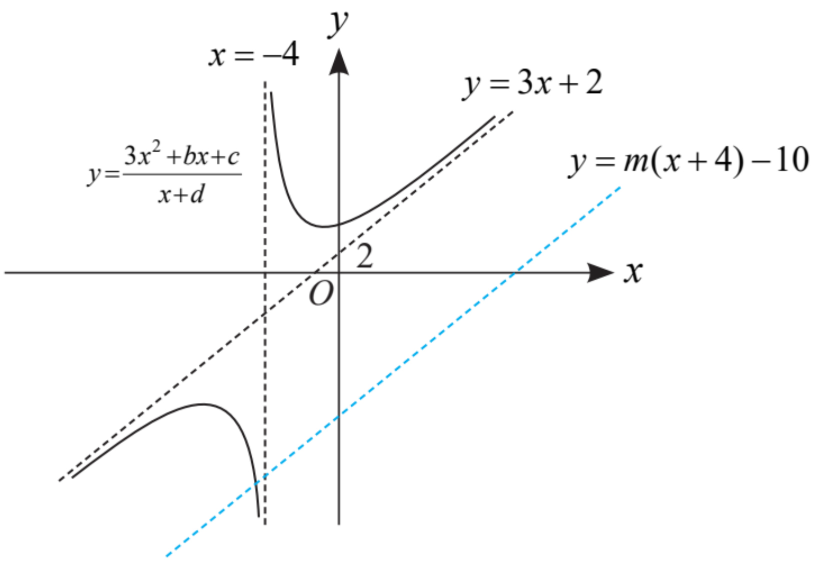
$y = -10,$ for all real values of m

Thus the point of intersection $(-4, -10)$ lies on the line $y = mx + 4m - 10$. (Verified)

Given $3x^2 - bx + c = m(-x+4)^2 + 10x - 40$

$$3x^2 - bx + c = (-x+4)\left[m(-x+4) - 10\right]$$

$$\frac{3x^2 - bx + c}{-x+4} = m(-x+4) - 10$$



To have 2 distinct roots for $\frac{3x^2 - bx + c}{-x+4} = m(-x+4) - 10$, that the curve $\frac{3x^2 + bx + c}{x+4}$ need to intersect $y = m(x+4) - 10$ twice.

$\therefore$ the range of values of m is $m > 3$.

49
Solution

(a) Given $\quad y = \dfrac{x^2 + 2x - 3}{x - 2}, x \neq 2$

$$xy - 2y = x^2 + 2x - 4$$

$x^2 + (2 - y)x + (2y - 4) = 0$ .... (1)

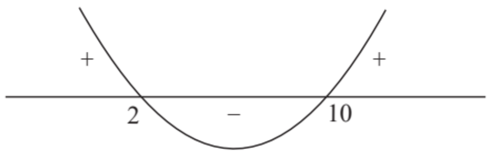
For (1) to have real roots of x, discriminant of $(1) \geq 0$

$(2 - y)^2 - 4(1)(2y - 4) \geq 0 \qquad \triangleleft$ using $b^2 - 4ac$

$4 - 4y + y^2 - 8y + 16 \geq 0$

$y^2 - 12y + 20 \geq 0$

$(y - 10)(y - 2) \geq 0$



$\therefore$ the range of values of y is $y \leq 2$ or $y \geq 10$

(b) Given $y = \dfrac{x^2 + 2x - 4}{x - 2}$

$$= x + 4 + \frac{4}{x - 2}$$

Equations of asymptotes

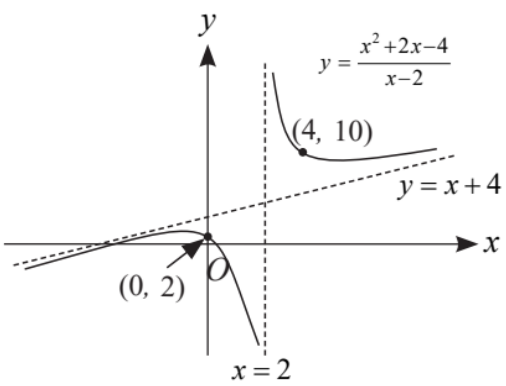
$x = 2$is a vertical asymptote.

$y = x + 4$ is an oblique asymptote.

Use G.C. to find turning point.

Minimum point $(1, 1)$ and Maximum point $(-2, -2)$.

The graph of $y = \dfrac{x^2 + 2x - 4}{x - 2}$



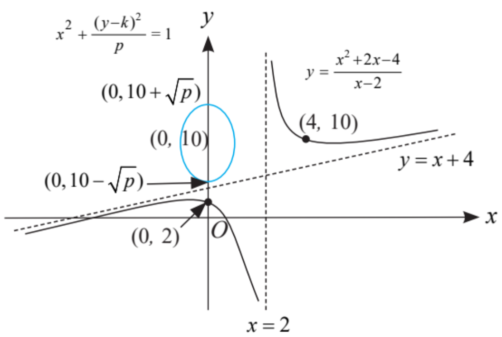
(c) $px^2 + y^2 020y + 100 - p = 0$

$$px^2 + (y-10)^2 = p$$

$$x^2 + \frac{(y-10)^2}{P} = 1 \quad \text{(Shown)}$$

The equation represents an ellipse with centre (0, 0) with length of semi - major axis $\sqrt{p}$ and length of semi - minor axis 1.

Add the graph $x^2 + \dfrac{(y-10)^2}{P} = 1$ on the same diagram.



(d) For C_1 and C_2 intersect at exactly two distinct points, $\sqrt{p} > 8$

i.e. $p > 64$

Given $p < 100$

$\therefore\ 64 < p < 100$

50
Solution

(a) Given $y = \frac{9}{t} - 1$

When $y = 0$,

$$0 = \frac{9}{t} - 1$$
$$t = 9$$

Substitute $t = 9$ into $x = 13 - \sqrt{t^2 + 144}$.

$\therefore\ x = -2$

$\therefore$ the coordinates that cuts the x-axis are $(-2, 0)$.

Given $x = 13 - \sqrt{t^2 + 144}$

When $x = 0$,

$$0 = 13 - \sqrt{t^2 + 144}$$
$$t = 5$$

Substitute $t = 5$ into $y = \frac{9}{t} - 1$.

$\therefore\ y = 0.8$

$\therefore$ the coordinates that cuts the y-axis are $(0, 0.8)$.

(b) $x = 13 - \sqrt{t^2 + 144}$ $\qquad$ $y = \frac{9}{t} - 1$

$$\frac{dx}{dt} = -\frac{1}{2}(t^2 + 144)^{-\frac{1}{2}}(2t) \qquad \frac{dy}{dt} = -\frac{9}{t^2}$$
$$= -\frac{t}{\sqrt{(t^2 + 144)}}$$

Using chain rule,

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{dy}{dt} \times \frac{dt}{dx}$$
$$\frac{dy}{dx} = \frac{9\sqrt{t^2 + 144}}{t^3}$$

For $t > 0$, $\sqrt{t^2 + 144} > 0$ and $t^3 > 0$

$\therefore\ \frac{dy}{dx} = \frac{9\sqrt{t^2 + 144}}{t^3} > 0$ for $t > 0$,

The curve has no stationary points. (Shown)

(c) **(i)** When $t \to 0, \sqrt{t^2 + 144} \to 12$

So, $13 - \sqrt{t^2 + 144} \to 1$

$\therefore\ x \to 1$ as $t \to 0$.

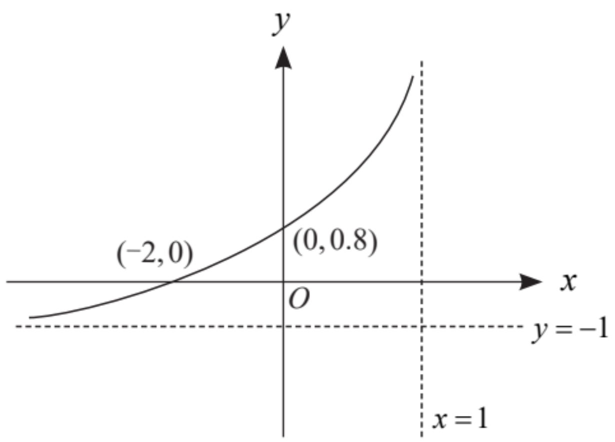
(ii) When $x \to -\infty,\ \dfrac{9}{t} \to 0$.

So, $\dfrac{9}{t} - 1 \to -1$.

$\therefore\ y \to -1$ as $x \to -\infty$.

(d) The graph of parametric equations

$$x = 13 - \sqrt{t^2 + 144},\ y = \frac{9}{t} - 1, \text{ where } t > 0.$$



51
Solution

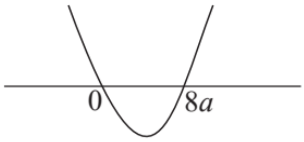
(a) Let $\dfrac{(2x-a)^2}{3(x-2a)}=k$

$4x^2-4ax+a^2=3kx-6ak$

$4x^2-(4a+3k)x+(6ak+a^2)=0$

For x to have no real solutions, $b^2-4ac<0$.

$$4(a+3k)^2-4(4)(6ak+a^2)<0$$
$$k^2-8ak<0$$
$$k(k-8a)<0$$
$$0<y<8a$$



Hence, the set of values that y cannot take is $\{y\in\mathbb{R}:\ 0<y<8a\}$

(b) When $a=1$, $\quad y=\dfrac{(2x-1)^2}{3(x-2)}$

$$=\frac{4x^2-4x+1}{3x-6}$$
$$=\frac{4}{3}x+\frac{4}{3}+\frac{3}{x-2}$$

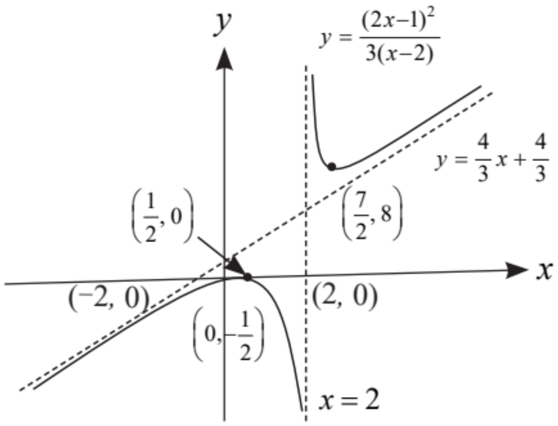
Equation of asymptotes : $x=2$ and $y=\dfrac{4}{3}x+\dfrac{4}{3}$

Intercepts : when $x=0$, $y=-\dfrac{1}{2}$

when $y=0$, $x=\dfrac{1}{2}$

From GC : Stationary Points $\left(\dfrac{1}{2},0\right)$ and $\left(\dfrac{7}{2},8\right)$

The graph of $y=\dfrac{(2x-1)^2}{3(x-2)}$



(c) Given $x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$ .... (1)

and $\quad y = \cos\theta$ .... (2)

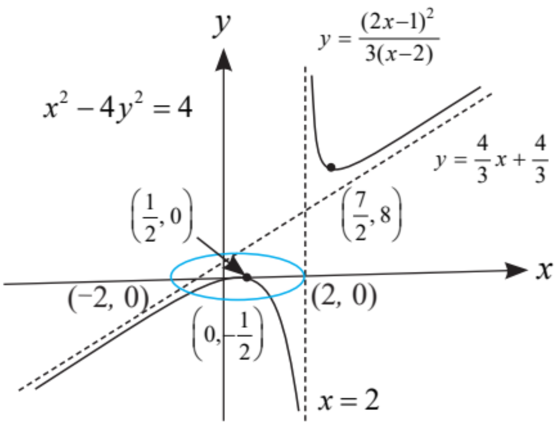
Using $\cos^2\theta + \sin^2\theta = 1$ .... (3)

Substituting (1) and (2) into (3)

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 1$$

The equation represents an ellipse with centre (0, 0) with length of semi - major axis 2 and length of semi - minor axis 1.



(d) From the above diagram, the curves intersects twice. Hence, there are 2 real roots to the equation.

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Solution

(a) Given $C: y = x - a + \dfrac{2-a}{x+a}$

The equations of the asymptotes are $y = x - a$ and $x = -a$.

(b) Solve $y = x - a$ and $x = -a$ simultaneously to find the point where the two asymptotes intersect,
$\therefore\ (-a, -2a)$.

Let $y = mx + a(m-2)$ .... (1)
Substituting $(-a, -2a)$ into (1).

$$\begin{aligned}\text{RHS} &= -ma + am - 2a, \text{ for all real values of } m \\ &= -2a \\ &= \text{LHS}\end{aligned}$$

Hence the line passes through the point $(-a, -2a)$. (Verified)

(c)(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \dfrac{2-a}{(x+a)^2}$

At stationary, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

i.e. $1 - \dfrac{2-a}{(x+a)^2} = 0$

$$\frac{2-a}{(x+a)^2} = 1$$

$$(x+a)^2 = 2 - a$$

$$x + a = \pm\sqrt{2-a} \quad \text{.......................... (2)}$$

x is undefined if $a > 2$

$\therefore$ the range of values of a for which C has no stationary point is $a > 2$.

(c)(ii) When $y = 0$,

$(x-a)(x+a) + (2-a) = 0$

$x^2 - a^2 + 2 - a = 0$

$x^2 = a^2 + a - 2$

$x^2 = (a+2)(a-1)$

For the values of x to exist, $(a+2)(a-1) > 0$

$\therefore\ a < -2$ or $a > 1$

$\therefore$ the range of values of a for which C cuts the x-axis at two distinct points is $a < -2$ or $a > 1$.

(d)(i) When $a = 4$,

$$y = x - 4 - \frac{2}{x+4}$$

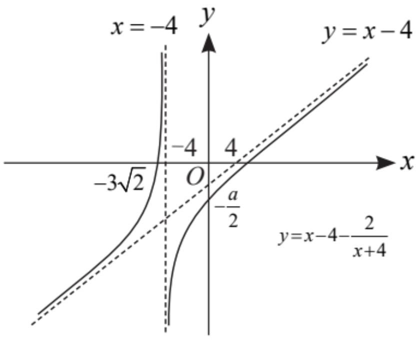
Equations of asymptotes
$x = -4$ is a vertical asymptote.
$y = x - 4$ is an oblique asymptote.

Axial intercept : When $x = 0$, $y = -\frac{a}{2}$

When $y = 0$, $x = -3\sqrt{2}$

From **(c)(i)**, C has no stationary point if $c > 2$.
In this case, where $c = 4$, $\therefore$ there is no stationary point.

The graph of $y = x - 4 - \frac{2}{x+4}$.



(d)(ii) When $1 < a < 0$,

$$y = x - a + \frac{2-a}{x+a}$$

Equations of asymptotes
$x = -a$ is a vertical asymptote.
$y = x - a$ is an oblique asymptote.

Axial intercept :
From **(c)(ii)**, C cuts the x-axis at two distinct points if $a < -2$ or $a > 1$.
In this case, where $1 < a < 0$, $\therefore$ there is no x-intercept.

When $x = 0$, $y = -a + \frac{2}{a} - 1$.

To find the stationary point, let $\dfrac{dy}{dx} = 0.$

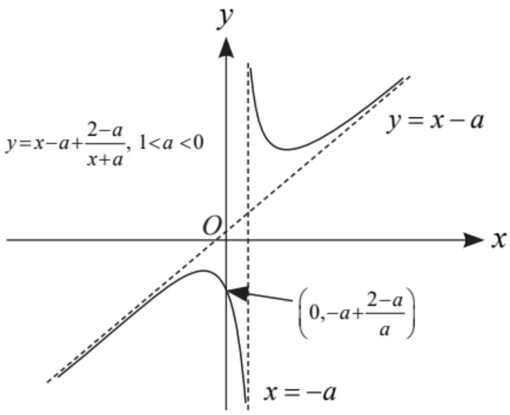
$\therefore \quad x = -a \pm \sqrt{2-a}$

$\dfrac{d^2y}{dx^2} = \dfrac{2-a}{(x+a)^3}$

When $x = -a - \sqrt{2-a}$, $\quad \dfrac{d^2y}{dx^2} < 0$ (maximum point).

When $x = -a + \sqrt{2-a}$ $\quad \dfrac{d^2y}{dx^2} > 0$ (minimum point).

The graph of $y = x - a + \dfrac{2-a}{x+a}$, where $1 < a < 0$.



(e)

For the line $y = mx + a(m-2)$ does not cut the curve C, the gradient of $y = mx + a(m-2)$ flatter than the gradient of the line $y = x - a$.

$\therefore \; m \leq 1$

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Solution

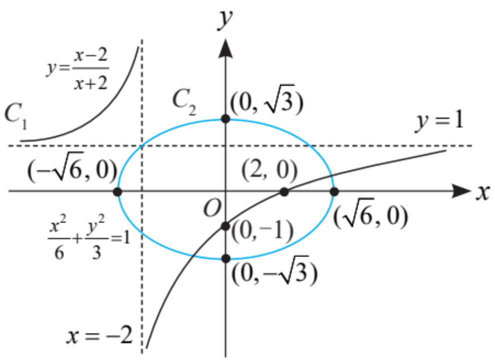
(a) $y=\dfrac{x-2}{x+2}$

$y=1-\dfrac{4}{x+2}$

Equations of asymptotes: $x=-2$ is a vertical asymptote and $y=1$ is a horizontal asymptote.

Axial intercept : When $x=0,\ y=-1$
When $y=0,\ x=2$

The graphs of $y=\dfrac{x-2}{x+2}$ and $\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$.



(b) Substitute $y=\dfrac{x-2}{x+2}$ into $\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$.

$$\frac{x^2}{6}+\frac{1}{3}\left(\frac{x-2}{x+2}\right)^2=1$$

$$x^2(x+2)^2+2(x-2)^2=6(x+2)^2$$

$$2(x-2)^2=(x+2)^2(6-x^2) \quad \text{(Shown)}$$

Use GC to solve the equation. $x=-0.505$ or $x=2.45$.

$\therefore$ x-coordinates are -0.505 and 2.45.

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Solution

(a) Given $t \geq 4$, $3+\sqrt{t} \geq 5$

$\therefore$ the range of values of x is $x \geq 5$.

Given $t \geq 4$, $-2+\sqrt{t-4}$

$\therefore$ the range of values of y is $y \geq -2$.

(b) Let $x = 3+\sqrt{t}$ .... (1)

From (1): $t = (x-3)^2$ .... (2)

Let $y = -2+\sqrt{t-4}$ .... (3)

From (3), $t = 4+(y+2)^2$ .... (4)

Equating (3) and (4)

$\therefore$ $(x-3)^2-(y+2)^2 = 4$ (Shown)

$y = -2 \pm \sqrt{(x-3)^2-4}$

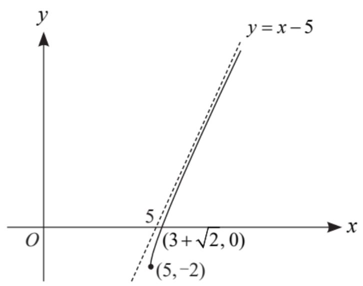
Since $y \geq -2$, $y = -2+\sqrt{(x-3)^2-4}$

Equation of the asymptote

$y+2 = x-3$

$y = x-5$

The graph of $y = -2+\sqrt{(x-3)^2-4}$, for $x \geq 5$ and $y \geq -2$.



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Solution

(a) Given $c^2x^2 - b^2y^2 - a^2 = 0$

Differentiating both sides with respect to x

$$2c^2x - 2b^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

i.e. $2c^2x = 0$

$\therefore\ x = 0.$

Substituting $x = 0$ into $c^2x^2 - b^2y^2 - a^2 = 0$

$\therefore\quad -b^2y^2 - a^2 = 0$

$y^2 = -\frac{a^2}{b^2}$ which is undefined since $y^2 \geq 0$.

$\therefore$ the curve has no turning point. (Shown)

(b) Given $c^2x^2 - b^2y^2 - a^2 = 0$

$$\frac{c^2x^2}{a^2} - \frac{b^2y^2}{a^2} = 1$$

$$\frac{x^2}{\left(\frac{a}{c}\right)^2} - \frac{y^2}{\left(\frac{a}{b}\right)^2} = 1$$

The equation represents a hyperbola centred at $(0, 0)$ and opens up and down. The gradient of the two equations asymptotes are $\pm\frac{c}{b}$.

Equations of the asymptotes are $y - 0 = \pm\frac{c}{b}(x - 0)$

$\therefore$ the equations of asymptotes are $y = \pm\frac{cx}{b}$.

Learning point:

Given equation of the hyperbola of the form $\frac{(x-h)^2}{a^2} + \frac{(x-k)^2}{b^2} = 1$,

the centre of the hyperbola is $(h,\ k)$ and the gradients of asymptotes are $\pm\frac{a}{b}$.

Equations of the asymptotes are $y - h = \pm\frac{a}{b}(x - k)$

(c) $y = \pm\dfrac{\sqrt{c^2x^2 - a^2}}{b}$

For y to be undefined, $c^2x^2 - a^2 < 0$.

i.e. $$x^2 - \frac{a^2}{c^2} < 0 \quad \triangleleft \text{ factorise}$$

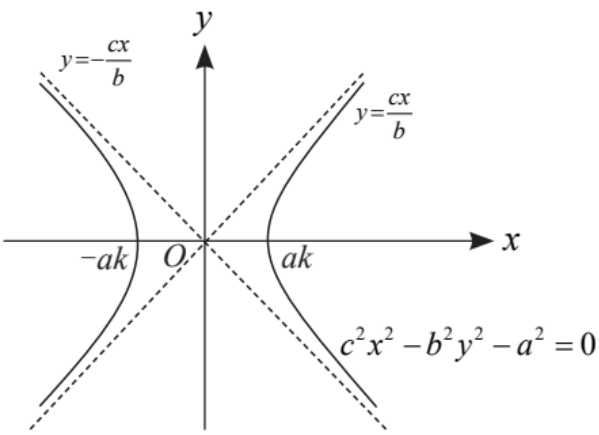
$$\left(x + \frac{a}{c}\right)\left(x - \frac{a}{c}\right) < 0$$

$$-\frac{a}{c} < x < \frac{a}{c}$$

$\therefore$ the restriction of x is $-\dfrac{a}{c} < x < \dfrac{a}{c}$.

The axes of symmetry of C : $y = 0$ and $x = 0$.

(d) The graph of curve C $c^2x^2 - b^2y^2 - a^2 = 0$



(e) Given $(c-bk)(c+bk)x^2 = a^2$

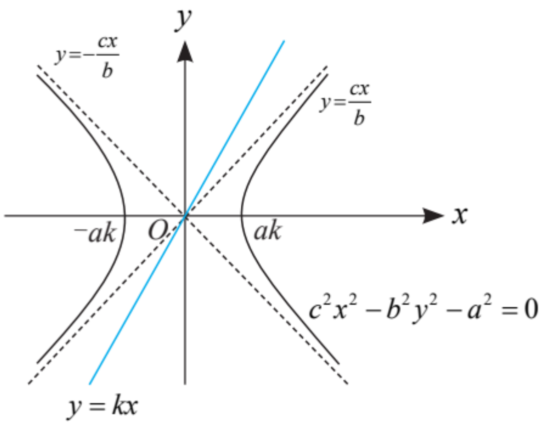
$(c^2-b^2k^2)x^2 = a^2$

$c^2x^2 - a^2 = b^2k^2x^2$

$\dfrac{c^2x^2-a^2}{b^2} = \left(kx\right)^2$

$\pm\dfrac{\sqrt{c^2x^2-a^2}}{b} = kx$

$\therefore$ the additional graph to draw $y = kx$.



For the line $y = kx$ intersects C twice, the gradient of the line needs to be steeper compared to the asymptotes of C

i.e. $k < k < \dfrac{c}{b}$ or $-\dfrac{c}{b} < k$.

$\therefore \quad -\dfrac{c}{b} < k < \dfrac{c}{b}$

Given $0 < c < b$, ◁ divide b on all sides

$$0 < \frac{c}{b} < 1$$

$-1 < -\dfrac{c}{b} < k < \dfrac{c}{b} < 1$

$\therefore \quad 1 < k < 1$ (Shown)